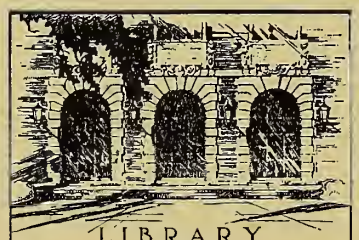

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PROPOSAL

FROM

UICSM

TO

NSF—CCI

DECEMBER 1, 1962

Submitted by: University of Illinois, Urbana, Illinois

Title: UICSM Project for the Improvement of School Mathematics

Project director: Professor Max Beberman, 1208 West Springfield, Urbana, Illinois; 217-333-0150

Dates: June 1, 1963 - August 31, 1964

Abstract: This proposal requests funds for the second year of the NSF-supported project for the development of curriculum materials in mathematics for grades 7-12.

Date of submission: December 1, 1962

F. T. Wall, Chairman
University Research Board

Max Beberman, Director
UICSM Project

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Major Objective of the University of Illinois Committee on School Mathematics

The University of Illinois maintains a secondary school [University High School] on its campus for one primary reason--to serve as a laboratory for the design and preliminary experimental trial of instructional materials. These materials are developed by subject matter experts on the University faculty and teaching specialists on the staff of the laboratory school.

The UICSM Project staff is a group of subject matter and teaching specialists attached to University High School. It has this central purpose--to create instructional materials in secondary school mathematics, and then to test them, first in University High School, and then in secondary schools generally throughout the country.

Summary of activities for the period June 1, 1962 to date.

(1) Staff. Most of the present staff was recruited during the period May 1 - August 31, 1962. It consists of college mathematicians and teachers, high school teachers, specialists in educational experimentation, and graduate students in mathematics, education, and psychology. [See Exhibit A for a description of the staff and for biographical sketches of the present and prospective staff.]

(2) Advisory Board. The Provost of the University has set up a national board representing mathematics, science, psychology, and education. The Board advises the project staff on all phases of its work. [See Exhibit B for the minutes of the October 6, 1962 meeting.]

(3) Programing Work. Unit 1 of the UICSM textbooks ["The Arithmetic of the Real Numbers"] was rewritten in self-instructional form [by Brown, Golden, Hoffmann, Kansky, McCoy, McInerney, Rundus, Szabo, Wills, and Zwoyer under the active supervision of Beberman and Vaughan] and is currently being used experimentally in 9 schools with 10 teachers and 580 students. These self-instructional materials occupy 18 volumes having a total of 1375 text pages and 324 worksheet pages.

Additional materials based on a portion of Unit 4 ["Graphs and Ordered Pairs"] were prepared in self-instructional form [by Wolfe] in 6 volumes [two alternative presentations of 3 volumes each] with a total of 700 pages and 120 worksheet pages. These materials are being used experimentally in 5 schools with 360 students.

Currently, the Unit 6 Appendix on Logic and the Unit 9 Appendix on Solid Geometry are being programed [by Hale and by Szabo].

[See Exhibit E for a description and sample of the programed materials from Units 1 and 4. See Exhibit C on objective studies for a description of the research being conducted on the experimental use of these programed materials.]

One of the purposes of the present programing activity is that of staff-training in programing techniques. It is our intention to convert much of our new curriculum material to self-instructional form. We have found that this type of "trial" of new materials, in which a skilled teacher-programer builds a type of detailed "lesson plan", is most useful in developing new teaching approaches and in revealing gaps in the presentation in the conventionally-written materials. Such information is useful to course-content writers even when the programed materials are not intended to be released for general classroom use.

Our experience in programing Unit 1 of the UICSM text has also uncovered techniques useful for presenting algebra to students in grades below the ninth grade. These techniques [as exemplified in Exhibit E] will be of great usefulness in our 7th grade course-content development work.

Critical reviews of the present programed materials have been written by various members of the staff. The materials are now being revised [by Hoffmann, Kansky, Rundus, Tremblay, and Zwoyer under the supervision of Beberman and Vaughan] on the basis of the staff reviews, the feedback from cooperating teachers and from a programed class of ten students conducted at University High School, the analysis [by Gilpin] of the data from nine topic tests [written by Gilpin] and the standardized Unit 1 test, and an analysis of worksheet data [being carried out by Wolfe].

The revised materials will be published in the spring by the UICSM and will be made available in single copies to mathematics teachers. Our purpose in doing this is not to promote the use of our self-instructional materials at the classroom level, but rather to make available to teachers the host of techniques which we invented while preparing the programed presentation which can be easily adapted to regular classroom teaching.

In the fall of 1963 Tremblay will coordinate the testing of the revised edition of the programed materials in several experimental centers. This trial will help us determine what sort of role, if any, may be appropriate for our self-instructional materials in the ordinary course of classroom instruction of children.

[This work on self-instructional materials is supported in part by the U. S. Office of Education through a grant to Beberman and Stolurow.]

(4) Objective Studies. Our staff is currently developing techniques which are widely applicable to the problems of curriculum evaluation. We interpret the term 'evaluation' as a series of activities [called 'objective studies'] which accompany every stage of the planning and development of a new curriculum, not merely as the administration and analysis of final examinations. [The staff has the part-time services of the statistician Tatsuoka and the consultant services of Cronbach.]

[References are made throughout this document to various objective studies in progress and being planned. For a summary of these studies and a statement of the principles which guide this phase of our work, see Exhibit C.]

Work-in-progress and Plans.

(1) 1963 UICSM-NSF Summer Institute.

Again in 1963, a portion of our staff plus invited staff will offer a summer institute for the training of junior and senior high school mathematics teachers. Over 300 participants will receive special training in both content and pedagogy in the UICSM program for grades 9-12. [We are receiving funds for the support of this work from the Institutes Section of NSF. We shall submit a proposal in May for a similar summer institute in 1964.]

The existence of a summer institute on our campus has important advantages for us in performing our primary job of developing new materials. Those of our staff who are writers during the academic year find opportunities in institute teaching to use the participants as a sounding board for new ideas which have arisen in course-content development work. This feedback can be used in judging the acceptability of such ideas to teachers and is vitally important in planning the teacher commentaries which will accompany the new instructional materials. Also, this contact with secondary school teachers provides a good basis for the recruitment of teachers who can serve as field testers of new materials. Finally, our summer institute employment of visiting staff brings to the project center some of the most competent teacher-trainers and high school teachers in the country. Some of these people will be among the first field testers. The informal interaction that they can maintain with the regular project staff on duty during the summer keeps them aware of our latest thinking.

Some of our writing and programing staff also handle courses in mathematics for teachers for the Academic Year Institute and for University Extension. [Among these are Hart, McCoy, Wills, and Zwoyer. They will be joined in 1963-64 by the prospective staff member Sanders.]

(2) Cooperation with the Coordinated Sciences Laboratory.

Four of our staff members have received part of their salaries from the University of Illinois Coordinated Sciences Laboratory's Project PLATO, a computer-based teaching system. These people are engaged in the writing of PLATO programs for use in teacher-training during the 1963 UICSM-NSF Summer Institute. Some of the programs will aim to clarify mathematical concepts, while others will illustrate techniques of presentation.

Easley and Wills are carrying out an objective study in cooperation with the PLATO group in which an investigation is being made of physiological indices of the intrinsic reinforcement which takes place when a student discovers a generalization as he works on a set of discovery exercises. [See Exhibit C for more details.]

Brown, Hoffmann, Wolfe, and Zwoyer, and the prospective staff member Mueller will give part of their time in 1963-64 to developing mathematics programs for PLATO. The people in charge of the PLATO Project are first-rate physicists, mathematicians, and engineers who have recently taken an intense interest in educational technology. It is important for people doing work in course-content improvement to be aware of the impending changes in educational technology so that their materials will be applicable in all types of schools in the next decade. The UICSM wishes to continue its experimentation with all forms of educational media. The immediate return from our involvement in the PLATO Project is the rapid and detailed feedback provided by the printout from the CDC 1604 computer which controls PLATO. Thus, short mathematical developments can be tested quickly and accurately.

(3) UICSM Film Project.

Beberman and Hendrix have started work in cooperation with The University of Illinois Motion Pictures Unit [Sims] on the improvement and augmentation of the existing series of UICSM teacher-training films for ninth grade algebra teachers. Several additional films of demonstration classes [taught by Wills] and a series of filmed content lectures [planned by Hendrix and delivered by Beberman] will be produced. The end product will be a series of about 50 half-hour films plus a detailed manual for their use [written by Hendrix]. Our aim is to produce a series which will be useful to teachers who have had no prior acquaintance with the UICSM curriculum. It is planned that this series plus the manual will be available for commercial distribution by June 1964. [A grant for this work has been received from the Course Content Improvement Section of NSF.]

In the meantime, we are continuing our practice of distributing our five copies of the original series plus observation-discussion guides to summer and academic year institute groups and to inservice workshops. These groups are supervised by instructors who are well-acquainted with the UICSM program and who use the series for the original purpose of providing instruction in pedagogy for teachers who are concurrently studying the UICSM course in ninth grade algebra.

One of the important by-products of the present film project is that it reestablishes our classroom studio together with a fully-equipped film-editing center. We expect to be able to utilize these facilities in subsequent years for the preparation of films for teacher-orientation purposes in connection with the field testing of new instructional materials.

(4) Objective Studies.

[Please refer to the aforementioned Exhibit C for a summary of ongoing and proposed objective studies.]

(5) Course Content Development.

We describe below our current work and plans for developing new instructional materials. For convenience, we shall describe separately matters of content, writing procedures, staff writing assignments, testing procedures, and production procedures and goals.

(a) Content.

Work on the new materials for grades 7-12 is now proceeding on two fronts--a junior high school sequence and a senior high school course in vector geometry.

(i) The junior high school sequence starts with a patterned review by Wirtz of arithmetic operations which is designed to focus the student's attention on the basic mathematical structure of elementary arithmetic. By 'patterned review', we mean an arrangement of arithmetic exercises which by reason of their order and placement force students into an awareness of basic generalizations. Moreover, the exercises have much intrinsic interest. The mathematical structure referred to includes the properties of commutativity, associativity, and distributivity; the duality between addition and multiplication; and the relationships between addition and subtraction and between multiplication and division. The patterns by which the results of later exercises can be predicted from the results of earlier ones, are to be discovered by the student. [For samples of such exercises, see Exhibit D1.]

The patterned review will lead into informal work with functions and the solution of equations and inequations on an intuitive basis. The pedagogical approaches will utilize some of the techniques invented in developing the previous UICSM courses and refined in the present self-instructional materials [see Exhibit E].

The 7th and 8th grade courses will also contain a systematic genetic development of the real number system starting with the finite cardinals. [See Exhibit D2 for details of the development by Vaughan and Phillips.] This work will be interspersed with the informal arithmetic-algebra sequence in such a way that the informal work will foreshadow the systematic development through the promotion of nonverbal awareness.

Another part of the junior high school work will consist of a sequence in informal plane and solid geometry for grades 7 through 9. In addition to serving as preparation for a more formal senior high school course in geometry, the informal geometry sequence will provide students with the geometric knowledge necessary for their high school studies in science and the practical arts.

We plan to include applications from all sciences and from "daily life". Following the suggestion of our Advisory Board, we shall experiment with the use of practical or quasi-practical problems as a source of new mathematical ideas in contrast with developments in which mathematics itself provides the motivation and practical problems come in only as applications.

(ii) We are developing a senior high school course based on a vector approach to Euclidean geometry, analytic geometry, and trigonometry. This course assumes the background in informal geometry which we are building into the junior high curriculum, and starts with a carefully-motivated formal development of the concept of a vector space. All of the content of conventional geometry can be developed by a consideration of points in space and translations (vectors) of points in space. [For details of this development by Vaughan, see Exhibit D3.]

There is a clear need in mathematics education today for experimentation with vector methods in secondary school mathematics. Vector methods are indispensable in the physical sciences and are being used increasingly in the social sciences. Secondly, a geometry course based on vectors gives a student an opportunity to employ his algebraic skills in proving theorems in geometry. [Although a coordinates approach to geometry also provides for the use of algebraic skills, the vector approach appears to be superior because the manipulations are concerned with geometric entities such as points and translations rather than with numerical representations of geometric entities.] Finally, a vector approach provides students with many opportunities to make useful applications of functions.

(b) Writing procedures.

We recognize two ways in which content ideas are initiated and then transformed into instructional materials. The staff mathematicians [together with individual consultants] may produce a first draft of a course, or portions of a course may be designed by a larger group of mathematicians in short-term work conferenc

(i) When a course is initiated by a staff mathematician, the instructional materials pass through four phases before the first classroom edition is ready for preliminary experimentation at University High School. Work is started by the mathematician who prepares a detailed account of the mathematics involved, including sample exercises. This draft is then reviewed by another writer or a consultant who also adds exercise materials. The revised and augmented second draft is then passed to two or three associate-writers who, in close

collaboration with the staff mathematicians, convert the draft into classroom materials. This third draft is then reviewed by the staff in two ways. One way is through conventional armchair criticism; the other way is through attempting to translate the materials into self-instructional form. This latter procedure is actually a kind of classroom trial in which an expert teacher uses the third draft as a basis for preparing his "lessons". He ponders the reactions of hypothetical students to the approached delineated in the draft; perforce, he invents alternative approaches and writes additional exercises. He is likely to uncover gaps in developmental exercise sequences. He can even test some of his hypotheses by checking his program with one or more trial students. Finally, the reviews and the self-instructional versions of the trial draft are returned to the staff mathematician and the associate-writers for production of the fourth draft which is then ready for classroom trial.

The programing step in the sequence described above has two important by-products. It may produce self-instructional materials for eventual use in schools if it turns out that this aspect of educational technology is significant in promoting learning. Also, it will provide us with the raw material for including in our teacher commentaries a description of various pedagogical techniques which are alternatives to the ones given in the text proper.

(ii) The Advisory Board has suggested a second writing procedure which will take advantage of the talents of many more mathematicians than our project could afford for full-time work. The Board has suggested that we hold one-week conferences during the academic year and one-month conferences during the summer.

We are now in the process of planning a 6-day conference to be held in February 1963 at the University of Illinois Conference Center in Allerton Park, Monticello, Illinois. This conference will consider the problems involved in relating applications to the development of a basic mathematics curriculum. [See Exhibit F for an announcement of the conference.] At the present writing, R. E. Bellman, S. P. Diliberto, F. Mosteller, H. O. Pollak, A. E. Ross, and F. J. Weyl have agreed to submit papers, and R. Z. Norman and C. R. Wylie, Jr. have been asked to give papers. Invitations to participate have been sent to A. A. Blank, C. R. De Prima, R. P. Dilworth, F. J. Dyson, G. E. Forsythe, B. Friedman, D. W. Hall, M. S. Klamkin, P. A. Lagerstrom, W. G. Lister, E. E. Moise, M. Newman, R. C. Taliaferro, and J. D. Williams.

Our writing staff will be present at this conference and will have ample opportunity to interact with and to learn from the guests. Also, as an aid to other curriculum groups, the proceedings will be published.

We shall also seek help from the conferees in planning a month-long work conference to be held in June 1963. We hope that some of the February participants would be willing to participate in the June conference and lead our staff in the actual writing of materials.

The outcomes of these two conferences will help us determine procedures for organizing conferences during the 1963-64 academic year.

(c) Staff writing assignments.

The writing staff consists of research mathematicians, college teachers, high school teachers of mathematics, and persons with advanced training in the sciences.

(i) The junior high school arithmetic-algebra sequence has been initiated by Vaughan with the assistance of Phillips. Wirtz is building foreshadowing-type exercises. The next-to-final classroom draft of the beginning portions of these materials will be developed in the spring of 1963 by Hart, McCoy, Phillips, and Wirtz. Starting in the summer or fall of 1963, they will be joined by the prospective staff member Gelder and a research mathematician. The programers who will work on the critique of this draft will be Wolfe and Zwoyer, and the prospective staff members Annett and Heilman.

(ii) Work on the junior high school informal geometry sequence will commence in the fall of 1963 under the supervision of the prospective staff research mathematician Fisher. He will work with the prospective staff members Marston, Rucker, and Sanders. Rundus and Tremblay and the prospective staff member Banks will handle the programing phase.

(iii) The senior high school vector geometry course has been initiated by Vaughan. Our consultant, Professor Dick Wick Hall of Harpur College, has consented to review and contribute to the first draft. Hart, Masalski, and Szabo are responsible for preparing the third draft. We are planning for Fisher to assume full responsibility for the third and fourth draft preparations on a full-time basis starting July 1963, and on a visiting consultant basis in the spring of 1963. Hoffmann and Wills and the prospective staff member Mueller will carry out the programing critique.

(iv) It is hoped that a sample of specific applications will emerge from the February conference mentioned above. Golden, Miyashiro, and Steere will then prepare classroom materials embodying these applications. Brown and Kansky will do the programing work. These materials will be tried out at the appropriate grade levels in our laboratory school for the purposes of judging student interest, gaining information on prerequisites, and developing effective pedagogical approaches.

As our staff amasses information on and gains experience in teaching applications, we should become increasingly adept at integrating the work on applications in the appropriate places in the basic mathematics sequence.

(d) Procedures for classroom testing of materials.

All instructional materials will be tested in classes at University High School. We shall use members of the writing staff as teachers of these trial classes. Also, these classes are completely open to visits from and even teaching by other writers. [For example, if Fisher joins us as a staff mathematician, he plans to work very closely with the classes using the vector geometry materials.] The classes will also be observed on a regular basis by Holmes, Horne, Johnson, Rollins, Ruether, and the prospective staff member Dilley, all under the supervision of Hendrix. Their observation reports will be sent to all of the writing staff and used as a basis for revision as well as substance for the preparation of teacher commentaries.

(i) The junior high school materials will be taught by Beberman and McCoy to two classes at the 7th grade level.

The senior high school vector geometry course will be taught by Masalski and Szabo to two classes. The teachers and observers will maintain careful records of those intuitive geometry concepts which the course appears to presuppose and which the students seem to lack. This information as well as accounts of the classroom approaches used in supplying these lacks will be transferred to writers of the junior high school informal geometry sequence. We plan to keep the classes in each pair about two weeks apart with respect to amount of materials covered. This will give us an opportunity to make changes in the materials in accordance with the reactions of the first class in the pair and try these modifications on the second class. In effect, we shall have two "crops" of experimental results in the same school year.

(ii) Gilpin and Brown will prepare tests to be used in the classroom trials of the new material. The test items will be based on statements of objectives prepared by the staff mathematicians and writers in cooperation with Gilpin and Brown. This will assure that the materials are evaluated in the light of the objectives admitted to by the initiating mathematicians. The test items will be prepared before the evaluators have seen the final draft of the classroom materials. Thus, we shall measure how well we have taught that content which logically belongs in the course, rather than just how well we have taught those topics which succeeded in getting into the classroom draft. We shall combine this information with subjective

evidence gathered by the classroom teachers and observers and thus determine the need for modification of classroom materials, or, if necessary, changes in the structure of the objectives. Although we do not intend to ignore subjective evidence in our future work, the evaluation procedure as described above will enable and require us to take explicit notice of any modifications we make in our objectives. Contributing to the usefulness of the information which we shall receive from this formal sort of evaluation, is the fact that our evaluators combine competence in educational measurement with training in mathematics and experience in teaching the UICSM program.

(e) Productions procedures and goals.

As in the past operations of the UICSM project, we shall do all of the production work required for the preliminary versions of the materials. Since we plan to utilize a variety of techniques such as color printing, film strips, and overhead projectuals, we shall need moderately elaborate layout, photographic, and duplicating equipment. Also, since we shall be working on a tight schedule necessitated by writing materials for a course while it is being taught, we must have complete control of production facilities. Although these matters tend to increase administrative responsibilities, we cannot contemplate any arrangement which would compel us to depend upon commercial agencies for our day-to-day requirements for finished classroom materials.

By June 1964 we shall have produced and tested in the classroom materials for the seventh grade in arithmetic, algebra, and informal geometry, materials for a year of instruction in high school vector geometry, and units of work dealing with applications. Also, preliminary drafts of arithmetic-algebra-geometry for the eighth grade will have been written--these will be tested at University High School in 1964-65 along with revisions of the materials for the seventh grade, for the vector geometry course, and revisions of the units on applications.

BUDGET

June 1, 1963 - August 31, 1964

Budget.

I. Personnel costs	average monthly rate	full-time equivalents	full-time equivalents for CCI	cost to CCI FY 63	FY 64	FY 65
A. Administration						
(1) Director [1]	1,970	1.00	0.67	-	15,699	2,617
(2) Assistant director [1]	1,109	1.00	0.18	-	2,374	396
(3) Executive assistant [1]	982	0.50	0.50	-	5,893	982
(4) Secretaries [2]	505	2.00	2.00	-	12,120	2,020
(5) Advisory Board	-	-	-	-	600	-
					36,686	6,015
B. Materials development						
(6) Chief mathematician [1]	1,526	1.00	1.00	-	18,318	3,053
(7) Staff mathematicians [2]	1,688	2.00	2.00	-	40,519	6,753
(8) Associate writers [11]	989	10.00	8.34	-	92,798	15,466
(9) Programers [12]	845	10.50	8.71	-	88,254	14,711
(10) Assistants [2]	563	1.00	1.00	-	6,754	1,123
(11) Consultants	-	-	-	-	2,700	-
(12) Typists [8]	380	8.00	8.00	-	36,480	6,080
					285,823	47,186
C. Production of materials						
(13) Production coordinator [1]	914	0.50	0.50	-	5,486	914
(14) Duplicating machine optrs. [2]	390	2.00	2.00	-	9,360	1,560
(15) Draftsmen-illustrators [2]	350	2.00	2.00	-	8,400	1,400
(16) Hourly workers	-	-	-	-	6,000	9,000
					29,246	12,874
D. Experimentation with materials						
(17) Research coordinator [1]	1,211	1.00	1.00	-	14,539	2,423
(18) Researchers [5]	811	4.00	3.37	-	36,818	6,135
(19) Assistants [5]	472	3.25	2.47	-	14,237	2,372
(20) Field consultants [7]	721	4.00	3.27	-	28,426	4,740
(21) Typists [3]	330	3.00	2.56	-	10,166	1,694
(22) Hourly workers	-	-	-	-	10,000	1,667
					114,186	19,031
E. Conferences						
(23) Honoraria				18,000	25,500	-
(24) Travel expenses				2,500	6,250	-
(25) Living expenses				2,700	3,510	-
(26) Staff expenses				4,146	4,998	-
				27,346	40,258	
F. Employee Benefits						
(27) Retirement [9.677%] and Workmen's Compensation [0.6%]				-	45,177	7,530

II. Operating costs	cost to CCI FY 63	FY 64	FY 65
G. Travel			
(28) Administrative	-	2,400	-
(29) Professional meetings	-	10,120	-
(30) Field visits	-	1,680	-
(31) Consultant visits	-	1,890	-
(32) Advisory Board travel	-	2,898	-
		18,988	
H. Supplies			
(33) Duplicating and binding	-	10,192	3,398
(34) Office supplies	-	1,500	375
(35) Art and photography	-	350	100
		12,042	3,873
I. Equipment			
(36) Maintenance and repairs	-	2,229	371
(37) Duplicating	-	8,900	-
(38) Photography	-	4,280	-
(39) Data processing	-	300	-
(40) Drafting equipment	-	600	-
(41) Stenographic equipment	-	8,889	-
(42) Experimental apparatus	-	5,000	-
(43) Office furniture	-	3,750	-
(44) Books and periodicals	-	429	71
		34,377	442
J. Communications			
(45) Telephone	-	7,886	1,314
(46) Postage	-	1,950	1,950
		9,836	3,264
K. Indirect costs			
(47) University overhead	4,102	93,993	15,032
L. Housing			
(48) Rent and maintenance		9,900	1,650
TOTAL REQUESTED FROM CCI	31,448	730,512	116,897
			878,857

Budget explanation.

I. Personnel costs

Notes:

(a) The Average Monthly Rate is an index of salary level. The AMR for an individual is one fourteenth of his full-time-equivalent salary for the period July 1, 1963 through August 31, 1964. The rate listed in the budget for each category is the average of the AMRs for the individuals in that category.

(b) All persons on the staff will be employed for a period of 14 months which includes 14/12 months vacation. Under the terms of this employment [University of Illinois Y contract], the staff does not take the usual academic recesses.

(c) A comparison of the bracketed numerals and the first column of full-time equivalents shows the extent to which the UICSM project utilizes part-time people. It also shows that some positions do not yet require full-time attention. The second full-time equivalents column when compared with the first shows the extent to which project personnel divide their interests between CCI work and other UICSM activities.

(d) All salary figures are estimates. Actual salaries will be determined for nonvisiting staff in accordance with established University procedures for project personnel. Salaries for visiting staff are based on the salaries they would receive from their home institutions plus not more than 25% in displacement allowance.

A. Administration

(1) The director [Beberman] serves as chief administrator, as coordinator of the writing groups, and as director of the summer institutes, the film project, and the programed instruction project [U.S.O.E.-supported]. Some of his salary comes from the University High School budget, some from the summer institute budgets, and the rest from CCI.

(2) The assistant director [Hale] serves as chairman of the mathematics department at University High School in which capacity he is responsible for scheduling classes, assigning teachers, and communicating with parents of children in experimental classes. His CCI-connected duties include serving as vice-director, hiring certain personnel, representing the project at professional meetings, and greeting visitors to the project center. He also serves as associate director of the summer institutes, a job which occupies most of his time during the spring semester and the summer. Most of his salary comes from the University High School budget and from the summer institute budgets.

(3) The executive assistant [Golden] spends half of his time in administrative work and the other half as an associate-writer. In his administrative capacity, he is now working with the University planning group on designing a building to house the entire staff; he investigates various types of equipment preparatory to making purchases; he works with the director in budget-planning and in general project coordination. Also, he represents the project at certain professional meetings.

(4) The administrative secretaries supervise the nonacademic personnel in two of the project buildings, keep accounting records, submit payroll information, prepare purchase orders, and handle routine correspondence.

(5) The Advisory Board consists of 8 people of whom 2 are staff members of the University. The other 6 receive honoraria at the rate of $50 per day for attendance. The Board meets twice each year. [See Exhibit B for a list of Board members.]

B. Materials development

(6) The chief mathematician [Vaughan] is the central figure in materials development. In addition to his responsibility for initiating and writing various courses, he serves the project as an advisor to the entire writing staff.

(7) The staff mathematicians are research mathematicians who have developed an interest in precollegiate mathematics education. Their role in the project is that of working with the chief mathematician in initiating courses, in writing materials with the help of the associate-writers, in supervising revision work, and in serving as mathematical advisors to the writing staff. [Fisher has agreed to consider filling one of these positions. We hope to fill the other position with a mathematician who has had considerable experience in applications.]

(8) The associate-writers represent a wide diversity in background and a strong common interest in the development of mathematically and pedagogically sound instructional materials. Among the present staff we have:

(a) Wirtz--author of a new elementary arithmetic series, formerly director of research and engineering of a large industrial firm, and a mathematics consultant to the Bucks County School System.

(b) Phillips--formerly arithmetic editor of a publishing firm, principal author of an elementary mathematics series, and a widely known lecturer at workshops and teachers conferences.

(c) Hart--member of the University mathematics faculty, teacher of special pedagogy courses for the mathematics department.

(d) Szabo--formerly a mathematics teacher in a large selective Detroit high school, and Advanced Placement teacher at University High School.

(e) Masalski--formerly a mathematics teacher in Greenwich, Conn. and a specialist in developing materials for the overhead projector.

(f) Golden--formerly a graduate student in physics at Illinois and Texas with research experience in solid state physics; experience in programed instruction and in the design of fully-automated teaching machines at the University of Illinois Training Research Laboratory.

(g) McCoy--formerly a high school mathematics teacher in central Illinois testing UICSM materials; for five years a teacher coordinator ["field consultant"] for the UICSM project working in cooperating schools throughout the United States; outstanding institute instructor at the Universities of Illinois and Hawaii.

Prospective staff members to be added to this group are Marston [mathematics head at The Principia, recent textbook author, summer institute teacher, and one of the most experienced UICSM teachers in the country], Rucker [junior high school teacher in California, presently completing an Academic year Institute at South Carolina], Sanders [formerly an AYI participant at Illinois, presently on the mathematics staff of a state college and engaged in teacher-training not only at the college but in surrounding school districts using UICSM materials], and Gelder [professor of mathematics at a state college in Washington, with a long-standing interest in teacher-training and curriculum improvement in undergraduate mathematics]. The salaries to be paid to these new people are based on their present salaries plus not more than a 25% dislocation allowance.

Approximately 75% of the salaries for the associate-writers will come from CCI funds. The rest will come from the mathematics department and summer institute budgets.

(9) Eight of the programers for the 1963-64 staff are on the present staff. We plan to add four more persons to the present programing group to bring the total to twelve. Since these people will also have institute teaching duties and research duties, the effective number of programers is somewhat less than nine. All of the programers have had extensive experience with the UICSM and other "modern" programs. Six of the group are graduates of Academic Year Institutes at Illinois, Notre Dame, and Stanford.

(10) The two assistants have backgrounds in physics and chemistry and will work with Golden in preparing units on applications.

(11) We have made good use of our consultants [Rosenbloom, Hall, and Dienes] during the present year. In addition to the new points of view they bring to the staff, the consultants also serve an important function in compelling the staff to present and defend its ideas to outsiders. [The consultant honorarium rate is $100 per day]

(12) The eight materials development typists are highly skilled mathematical typists who can read mathematical symbolism and who can exercise artistic judgment in designing a page of text materials.

C. Production of materials

(13) We are most fortunate in having on our staff a person [Rundus] who is not only a UICSM-experienced teacher but who has had vast experience in photography and graphic design. Since he is also one of the programers, we have in him a substantial link between content development and the important problem of faithful execution of ideas through the physical layout and production of text materials.

(14) The project is preparing to purchase an offset duplicator. This machine requires a highly skilled operator. In view of our production plans, it is likely that we shall need to keep the machine in operation over 12 hours per day; hence the need for two operators.

(15) The programed instruction materials production we carried out last summer kept two persons fully occupied with illustration work. We estimate that two full-time draftsmen--assisted by hourly workers during peak periods--will be able to handle our illustration work.

(16) The hourly workers are used during peak production periods. We expect a very high peak during the summer of 1964 [fiscal year 1965]. Hourly workers do such things as punch paper, collate and bind books, wrap shipments of books, assist in the operation of the duplicating machines, proofread, assist the draftsmen, and in the case of those student-workers with a mathematical background, check answers to exercises.

D. Experimention with materials

(17) The research coordinator [Easley] joined our staff this year from his position on the science faculty of the University of Hawaii. He has had abiding interests in the philosophy of science and its relation to education, in the UICSM program and in the curriculum study groups in other fields, and in experimentation in the training of teachers. His role with us is primarily that of a researcher; however, he also has certain administrative responsibilities including the hiring of hourly workers who work on data reduction, the assignment of observers to the experimental classes, and the coordination of the many research activities of our objective studies staff.

(18) Although we do not plan any increase in the size of the research staff, we do expect an increase in the total amount of activity since many of the projects which are just starting this year (1962-63) will be in full swing next year. [See Exhibit C.] Two of our researchers [Tatsuoaka and Van Horn] are experts in psychometrics, one of them [Gilpin] has done research work in self-instruction for the Earlham College Project and at Bell Telephone Laboratories, and two of them [Brown and Tremblay] have had extensive experience in teaching the UICSM program.

(19) The assistants serve as "foremen" for the hourly workers in data reduction, and they also take responsibility for the more complicated aspects of this work. All of the assistants have had undergraduate majors in mathematics and three of them are graduate students in mathematics.

(20) Except for Hendrix, all of the field consultants are actually field-consultants-in-training. Their job in 1963-64 will consist of making systematic observations of the experimental classes at University High School. Their reports will be used in revising the new materials and in preparing the teachers commentaries. Their experience in reporting observations will be useful in setting up procedures for the field-testing which should start on a limited scale in 1964-65 and become much more widespread in 1965-66.

(21) The typists will work on observation reports, research reports, and test items. They will also handle small-scale duplicating machines for preliminary drafts of various papers.

(22) The hourly workers will be used in the mammoth job of data reduction.

E. Conferences

In line with the suggestion of our Advisory Board, we are planning a February 1963 conference [see part b(ii) on page 9] on the applications of mathematics and their role in precollege mathematics instruction. Although our 1962-63 budget does not provide specifically for the February 1963 meeting, we shall be able to transfer some funds from other parts of that budget. Such a transfer will not be possible for the

June 1963 conference; hence our present request for funds for that conference [fiscal year 1963].

We also anticipate a short academic year conference in 1963-64 and a longer one in June 1964. Thus, the proposed budget requests funds for two long conferences and one short one. Here is how we have reckoned costs for each type of conference:

Short conference

[15 visitors, 15 staff, 1 secretary, 6 days]

Honoraria:	$500 per visitor	$ 7500
Travel expenses:	$250 per visitor	3750
Living expenses:	$54 per visitor	810
Staff expenses:	$54 per staff member $12 per secretary	822
		$12,882

Long conference

[10 visitors, 15 staff, 2 secretaries, 30 days]

Honoraria:	$1800 per visitor	$ 18000
Travel expenses:	$250 per visitor	2500
Living expenses:	$270 per visitor	2700
Staff expenses:	$270 per staff $48 per secretary	4176
		$27,376

F. Employee benefits

(27) Retirement contributions are made for all regular staff above the rank of graduate assistant, and for full-time graduate assistants and all nonacademic employees except hourly workers. Workmen's Compensation contributions are made for all staff members.

G. Travel

(28) Administrative travel

This category includes travel to professional meetings at which

the Project should be represented, travel to policy-making conferences related to the work of the Project, and visits to schools and colleges comtemplating cooperation with the Project.

We estimate 12 trips at $200 each [$150 for transportation and $50 for 2.5 days of living expenses].

(29) Professional meetings

Our staff is constantly called upon for contributions to professional meetings. In some cases expenses are paid by the sponsoring groups, but in the case of larger organizations, and in the case of very small groups, the project provides the expenses.

We estimate 44 trips to professional meetings at $230 per trip [$150 for transportation and $80 for 4 days of living expense].

(30) Field visits

These are visits to be made in connection with the field tryout of programed materials [Unit 1 and Solid Geometry].

We estimate four 2-person visits of three day's duration with transportation expenses of $150 per trip per person and living expenses of $60 per trip per person.

(31) Consultant visits

This category covers travel and living expenses for the individual consultants mentioned in (11).

We estimate nine 3-day trips with transportation expenses of $150 and living expenses of $60 per trip.

(32) Advisory Board Travel

This category covers travel for the advisory board members and staff. [See (5).]

We estimate two 1-day meetings per year with 6 non-University people and 12 University people in attendance.

Non-University people: $175 transportation and
$17.50 living expenses per person

University people: $20 transportation per person

Lunches: $3.00 per person

H. Supplies

(33) Duplicating, binding, and packaging

We estimate that we shall need $6600 for offset masters made photographically at an average cost of $0.90 each. We shall need $1000 for ink and related machine supplies, $4200 worth of paper, $2000 worth of plastic binders, and $450 for cartons.

(34) Office supplies

We estimate $25 worth for each of 75 people.

(35) Art and photographic supplies

We estimate $200 for art supplies such as vellum, drawing ink, special pens and brushes, Zip-a-tone, etc., $50 for 35 mm. film stock for film strips in connection with objective studies, and $200 for overhead projector supplies. [The cost of a substantial amount of photographic supplies is included in the cost of development of offset masters listed in (33).]

I. Equipment

(36) Maintenance and repairs

We need maintenance contracts on 16 typewriters, 10 dictation machines, 1 offset machine, 4 mimeographs, 1 electronic stencil cutter, 2 power punches and binders.

(37) Duplicating equipment

This includes an offset camera [$1800], a paper-folding machine [$300], a power punch and binder [$1500], and a second offset machine [$5000].

(38) Photographic equipment

The University has converted one of our offices into a photographic laboratory [plumbing, and special lighting]. We need $2620 for darkroom equipment, $785 for a combination 35 mm and 4-by-5 copy camera, $75 for a registration punch, $375 for a stripping table, and $425 for an enlarger plus lenses.

(39) Data processing

We estimate $300 for computer, tabulation, and card-sorting services.

(40) Drafting equipment

We need $600 for 2 drafting tables equipped with drafting machines.

(41) Stenographic equipment

Although the UICSM Project has a considerable investment in special-keyboard typewriters, two of these are 5 or more years old and should be replaced. We estimate that we shall need $4600 for 7 special-keyboard electric typewriters, and $3000 for a Varityper for special typography. We also need $600 for 2 portable dictation machines and $689 for 3 transcription tape recorders.

(42) Experimental apparatus

We estimate $5000 for a Perceptoscope and attachments, control circuitry, data collection equipment, and student-station equipment to be used in an objective study. [See Exhibit C.]

(43) Office furniture

We estimate $250 in office furniture for each of 15 people.

(44) Books and periodicals

These include current texts, popularizations of science and mathematics and professional periodicals and catalogs.

J. Communications

(45) Telephone

Our basic telephone charges will be $7200 and we anticipate $2000 in long distance charges.

(46) Postage

We estimate $900 in ordinary mailings and $3000 in materials shipments.

K. Indirect costs

(47) We are requesting the maximum overhead allowed by the CCI section. This is 15% of the cost of all items (1) through (47).

L. Housing

(48) The existing UICSM facilities are insufficient to house the entire staff. We estimate we will need 3300 sq. ft. of additional space, and that space can be obtained and maintained for $3.00 per sq. ft. per year.

EXHIBITS A-F

TO ACCOMPANY THE DECEMBER 1, 1962 PROPOSAL

FROM

UICSM

TO

NSF—CCI

Exhibit A

Staff Description

A. Administrative staff

Director--	Max Beberman
Assistant director--	William T. Hale
Executive assistant--	William M. Golden

B. Writing staff

Chief Mathematician--	Herbert E. Vaughan
Staff Mathematician--	*Robert C. Fisher
Associate writers--	*Harvey M. Gelder
	William M. Golden
	Alice G. Hart
	*Howard Marston
	William J. Masalski
	M. Eleanor McCoy
	Josephine McK. Phillips
	*Walter E. Rucker
	*Walter J. Sanders
	Steven Szabo
	Robert W. Wirtz
Programers--	*William J. Annett
	*Wilson P. Banks
	O. Robert Brown, Jr.
	*Bruce E. Heilman
	Joseph R. Hoffmann
	Robert J. Kansky
	Robert E. Rundus
	Clifford W. Tremblay
	Herbert Wills
	Martin S. Wolfe
	Russell E. Zwoyer
Assistants--	Jane Y. Miyashiro
	Robin C. Steere

C. Production staff

Production head--	Robert Rundus

D. Experimentation staff

Research coordinator--	John A. Easley, Jr.
Researchers--	O. Robert Brown, Jr.
	John B. Gilpin
	Maurice M. Tatsuoaka
	Clifford W. Tremblay
	Charles J. Van Horn
Assistants--	Judith E. Boyle
	Patricia P. Cutler
	Edwin L. Dolney
	James F. McInerney
	Kukmi K. Pak
Field consultants--	Gertrude Hendrix
	*Clyde Dilley
	Allen H. Holmes
	Edgar B. Horne
	Robert L. Johnson
	Nancy L. Reuther
	James H. Rollins

* Prospective staff members

Biographical Sketches of 1963-64 Staff

William G. Annett

Born in Seaford, New York, in 1931. BS [mathematics], College of William and Mary, 1953; EdM, Hofstra College, New York, 1961. Teaching mathematics at Seaford High School, Seaford, New York, 1956- ; part-time instructor in mathematics at Nassau Community College, Mineola, New York, 1961- ; taught UICSM in-service program at Seaford High School, 1961-62. Participant in NSF-UICSM summer institutes, 1960, 1961, 1962. Publication: "Products of Signed Quantities-An Inductive Approach" THE MATHEMATICS TEACHER, March, 1961. Membership in New York State Math Teachers Association, Nassau County Math Teachers Association.

Wilson P. Banks

Born in Spring Arbor, Michigan, in 1930. BA [mathematics], Greenville College [Greenville, Illinois], 1952; MS [teaching of math] University of Illinois, 1959; Graduate work [mathematics] State University of Iowa, Ball State Teachers College and University of Michigan. Head of Mathematics Department at LeClaire, Iowa, 1954-57. Taught at Davenport, Iowa. [teaching UICSM], 1959-61. Mathematics Consultant to 15 school districts and writer on Greater Cleveland Mathematics Program, 1961-62. Mathematician at Rock Island [Illinois] Arsenal, 1956, and M. A. Ford Manufacturing Company, 1961. Writer for Iowa State Department of Public Instruction Curriculum Revision Committee, 1960-61. Member MAA, NCTM.

Max Beberman

Born in New York City in 1925. BS [mathematics] from City College in 1944, AM and EdD [teaching of mathematics] in 1949 and 1953, respectively, from Teachers College, Columbia University. [Dissertation: The Teaching of Statistics in Secondary School Mathematics.] Taught mathematics and science in Nome [Alaska] High School from 1946-48; taught mathematics in 1949-50 at Teachers College and at Riverdale Country Day School in New York City. Instructor in education and teacher of mathematics in the University of Illinois High School in 1950-53; assistant professor of education 1953-54; associate professor of education [Florida State University] 1954-55; associate professor of education [University of Illinois] 1955-58; Professor of education [University of Illinois] 1958- . Chairman, University of Illinois Committee on School Mathematics 1951-54; Director, UICSM Project 1955- . Author [with H. F. Fehr and W. H. Carahan] of Algebra, Course I and Algebra, Course II (published by D. C. Heath and Company, 1955): [with H. E. Vaughan and UICSM Staff] High School Mathematics Units 1-11, University of Illinois Press; Inglis Lecturer, Harvard University, 1958 [An Emerging Program of Secondary School Mathematics, Harvard University Press, 1959]. Publications: [with B. E. Meserve] "The Concept of a Literal Number Symbol",

The Mathematics Teacher, October, 1955; [with B. E. Meserve] "An Exploritory Approach to Solving Equations", The Mathematics Teacher, January, 1956; [with B. E. Meserve] "Graphing in Elementary Algebra", The Mathematics Teacher, April, 1956; "A Modern Mathematics Program", Minnesota Mathematics Newsletter, November, 1956; [with UICSM staff] "Words, 'Words', "Words" ". The Mathematics Teacher, March 1957, [with UICSM staff] "Arithmetic With Frames", The Arithmetic Teacher, April, 1957; "The Illinois School Mathematics Program", The School Review, Winter, 1957; "Improving High School Mathematics Teaching", Educational Leadership, December 1959. Member of the Board of Directors of the National Council of Teachers of Mathematics; membership in AMS, MAA, Phi Delta Kappa, AAAS, CASMT.

Judith E. Boyle

Born Judith Broderick in Freeport, Illinois, in 1938. BA [mathematics], Marycrest College [Davenport, Iowa], 1960. Taught Mathematics and English at Columbus High School, Waterloo, Iowa, 1960-61. Taught mathematics at Unity High School, Tolono, Illinois, 1961-62. Research assistant at the University of Illinois on the UICSM Mathematics Project, 1962- Member of ISEA.

O. Robert Brown, Jr.

Born in Berwyn, Illinois, in 1937. BA [mathematics] from Oberlin College, 1959; MA [mathematics] in 1961 from University of Illinois. Teacher of mathematics at University of Illinois High School, 1959- ; evaluator for UICSM Project 1959- ; Membership in MAA, NCTM.

Patricia P. Cutler

Born in London, England, in 1936. Attended University of Bristol [England] 1954-58; BS [mathematics and physics] 1957, Certificate in Education, 1958. Taught at Peckham School [girls comprehensive, London] 1958-62. Research assistant, UICSM Project, 1962- . Membership in Association of Assistant Mistresses, Association of Women Science Teachers.

Clyde Dilley

Born in Durand, Wisconsin, in 1931. BA [mathematics and science], Iowa State Teachers College, 1953; MA [mathematics] Northwestern University, 1959; attended UICSM summer institutes 1959-62. Taught at Franklin High School, Cedar Rapids, Iowa, 1955-57; York High School, Elmhurst, Illinois, 1957-59; Willowbrook High School, Villa Park, Illinois, 1959- . Member of NCTM, ICTM.

Edwin L. Dolney

Born in Webster, South Dakota, in 1921. BS [education] Northern State Teachers College [South Dakota], 1958; MS [education] University of Notre Dame, 1950; completing requirements for Ed.D. [mathematics education] University of Illinois [estimate 1964]. Taught at University of Alaska, 1950-55; University of Missouri School of Mines and Metallurgy, 1955-56; Eastern Montana College, 1956-62 [Leave of absence, Assoc. Prof. of Mathematics]. Participant, University of Illinois NSF Mathematics Institute, 1959-60. Lecturer, NSF in-service mathematics institutes at Eastern Montana College. Research Associate, UICSM Project, 1962- . Membership in NCTM, MAA, AAUP.

John A. Easley, Jr.

Born in Manning, South Carolina in 1922. BS [physics] Wake Forest College [North Carolina], 1943; Graduate work [physics] University of California, 1946; MEd [science education] University of Hawaii, 1952; PhD [science education] Harvard University, 1955; post doctoral study of curriculum projects in science and mathematics [University of Illinois] 1959-60. Radio engineer for Carnegie Institute of Washington in Washington, D. C., Baffin Is., and Hawaii, 1943-46; Physics instructor Vallejo Junior College, California, 1947; Principal, Marshall Islands Intermediate School, Magino, Marshall Islands, 1949-50; Instructor in Science Education, University of Hawaii, 1950-52; Teaching Fellow in education, Harvard University, 1952-54; Assistant Professor of Science, University of Hawaii [Hilo Campus] 1955-60. Associate Professor, University of Hawaii [Hilo], 1960- , [on leave 1962-63]; Visiting Associate Professor, UICSM Project 1962- . Peace Corps [consultant in Science and Mathematics education], Philippines and in California, 1961-62. Publications: "A Study of Scientific Method as an Educational Objective", [PhD dissertation], 1955, Widener Library, Harvard University; "A Pedagogical Device for Clarifying the Concept of Drift", Proceedings of the Hawaiian Academy of Science, 1957; "Is Scientific Method a Significant Educational Objective?", Philosophy and Education [Israel Scheffler, editor], Allyn and Bacon, 1958; "The Physical Science Study Committee and Educational Theory", Harvard Educational Review, Winter, 1959; "The Challenge for Teacher Education of New Curriculum Projects", University of Hawaii Symposium of Education, 1960; "A Teaching Laboratory, The New Curriculum Materials and a Liberal Education", [presented AAAS meeting in Denver, December, 1961], Abstract in Journal of Teacher Education, June, 1962; [with J. Miyashiro] "Laboratory Courses in the Art of Teaching--an Exploratory Study", research supported by AAAS Studies in Teacher Education, September, 1962. [to be presented in condensed form at AERA meeting in Chicago, February, 1963]; [with M. M. Tatsuoka], Introduction to Scientific Thought in the Physical Sciences, [a college level general science text], Wadsworth Publishing Co. [in press]. Membership in Phi Kappa Phi, Omicron Delta Kappa, Phi Delta Kappa, AAUP, Hawaiian Academy of Science, AAAS [fellow], John Dewey Society [fellow], Philosophy of Education Society.

Robert C. Fisher

Born in Shelbyville, Indiana, in 1926. PhD [mathematics], University of Kansas, 1952. Since 1952, in the Department of Mathematics at the Ohio State University, presently an associate professor. Director, NSF Summer Institutes in Mathematics at Ohio State University, 1960 and 1961. Distinguished Teaching Award from Ohio State University alumni association, 1960. Co-author two books, Algebra and Trigonometry, Calculus and Analytic Geometry. Member AMS, MAA, NCTM.

Harvey M. Gelder

Born in Greeley, Colorado, 1916. AB [mathematics] in 1938, Colorado State College; MA [mathematics] in 1948 from the University of Missouri. Further graduate work, University of Chicago, University of Washington. Taught at Highland Park [Illinois] High School, 1938; Grand Junction [Colorado] High School, 1939-41; Kemper School [Missouri], 1941-43 and 1945-47. Now Associate Professor [mathematics], Western Washington State College; Chairman, Department of Mathematics, 1952-62. Member, Committee on Secondary Lecturers, MAA; Advisory Committee on Mathematics and Advisory Committee on Certification, State of Washington. Officer, Pacific Northwest Section, MAA. Member NCTM, AAUP.

John B. Gilpin

Born in Carbondale, Pennsylvania, in 1937. Attended Dickinson College 1954-58, with courses also from the Carnegie Institute of Technology and the Pennsylvania State University, BS [mathematics], 1958. Worked with the Teaching Machines Research Group at Harvard University, 1958, the Instructional Processes Section of Bell Telephone Laboratories [Murray Hill] 1958-59, the Earlham College Self-Instruction Project, 1959-62; Research Associate in Education at the University of Illinois beginning August, 1962. Publications: "A Versatile Apparatus for Audio Self-Instruction", Programing of Audio-Lingual Language Skills for Self-Instructional Presentation, University of Michigan, 1961; "The Design and Evaluation of Instructional Systems", Audiovisual Communication Review, March-April, 1961; "The Specification of Instructional Objectives", paper presented to NEA-DAVI, Kansas City, March, 1962. Member of NEA-DAVI, AAAS, and the American Society of Training Directors.

William M. Golden

Born in Chicago, Illinois, in 1938. BS [engineering physics], University of Illinois, 1959; MS [physics], University of Illinois, 1961; graduate work [physics], University of Texas, 1961-62. Laboratory assistant, University of Illinois, Department of Physics Radiation Damage Group, 1959-61. Research in programed instruction and automatic teaching devices, University of Illinois Training Research Laboratory, 1961; UICSM Programed Instruction Project, summer 1961; Research Associate UICSM staff 1962- . Member of AAAS.

William T. Hale

Born in Monson, Massachusetts, in 1928. AB [mathematics] in 1953 from Bates College; EdM in 1954 from University of Illinois; completing requirements for PhD in mathematics education at University of Illinois [est. 1963]. Taught at Lowville [New York] Academy and Central School, 1954-56; University of Illinois High School, 1956- . Assistant Director, UICSM Project, 1957- . Taught at NSF summer institute at University of Kansas, Syracuse University, Wayne State University, University of Illinois. Publication: "UICSM's Decade of Experimentation", The Mathematics Teacher, December, 1961. Member of Phi Beta Kappa.

Alice G. Hart

Born in Columbus, Kansas, in 1912. BA and BS in mathematics and BS in education in 1930 from Central Missouri State Teachers College. Graduate work in mathematics at Louisiana State University and University of Missouri. MS [mathematics], University of Illinois, 1959. Taught high school mathematics in Madison, Missouri, in 1930-31, and in West Plains, Missouri, in 1931-38. Taught high school and junior college mathematics in Jefferson City, Missouri, 1938-43. Taught high school mathematics in Tacoma, Washington, 1955-58; teacher associate with UICSM Project 1958-60; assistant professor of mathematics [Academic Year Institute and UICSM] 1960- . Cryptanalyst with Military Intelligence 1943-44, office manager and accountant 1949-55. Publication: [with M. Beberman] "New Approach to Mathematics", Discovery Magazine, March, 1962.

Bruce E. Heilman

Born in Evanston, Illinois, in 1928. AB [mathematics-economics] in 1952 from Hobart College, graduate study in mathematics at Arizona State University, 1956-58, San Diego State College, 1960-61. Attended NSF Summer Institutes Northwestern University 1960, University of Illinois, 1961 and 1962. Graduate Teaching Assistant [mathematics] Arizona State University, 1957-58. Taught secondary school mathematics, Brentwood Public Schools, New York, 1958-59, 1961-63, Dysart School, Peoria, Arizona, 1959-60. Director, Mathematics In-Service Teacher Training Program, Brentwood Public Schools, 1961-63. Member NCTM, CASMT, NYSTA, Suffolk County Mathematics Teachers Association.

Gertrude Hendrix

Born in Putnamville, Indiana, in 1905. AB [mathematics] in 1926 from DePauw University; MS [education] from University of Illinois in 1930; MA [mathematics] in 1935 from University of Illinois; unclassified graduate work in formal logic, philosophy of science, foundations of mathematics, psychology of learning, and theory of signs in Summers 1940, 1941, and academic year 1946-47 at Univsity of Chicago. Taught at Crossville [Illinois] Community High School 1926-38, principal February-May, 1928; University of Illinois High School, 1928-

30; Eastern Illinois University, 1930-56; taught mathematics at DePauw University, Summer 1944, and at University of Illinois, Summer 1946; UICSM Project 1956- ; Visiting Assistant Professor of Education, University of Illinois, 1958-62; Research Associate Professor of Education, University of Illinois, 1962- ; present position: Teacher Coordinator and Field Consultant, UICSM; Content Director, UICSM Film Project, Principal Investigator, University of Illinois Nonverbal Awareness Study. [Avocation: breeding, training, showing American Saddle Horses and teaching equitation, 1937-1956. Normal Course Certificate from the Teela-Wooket School of Equitation, Roxbury, Vermont. Summer staff, Teela-Wooket Riding Camps and School of Equitation, 1937-39, 1943. Operator, Polycreek Stable, Cloverdale, Indiana, Summers, 1946-55; owner and operator, Polycreek Farm 1961- .] Publications: "Variable Paradox--A Dialog in One Act", School Science and Mathematics, June, 1959. "Learning by Discovery", The Mathematics Teacher, May, 1959; "The Psychological Appeal of Deductive Proof", The Mathematics Teacher, November, 1961. Teaching Devices on the High School Level, University of Illinois Bureau of Educational Research, 1932. [with H. C. Barber] Plane Geometry and Its Reasoning, Harcourt-Brace, 1937; Editor, ICTM News Letter, 12-page quarterly, 1956-61; "A Protest Against Informal Reasoning as an Approach to Demonstrative Geometry", Mathematics Teacher, April, 1936; "...Mathematical Knowledge and Abilities for the Teacher of Geometry... in Fields Other Than Geometry", Mathematics Teacher, February, 1941; "A New Clue to Transfer of Training", Elementary School Journal, December, 1947; "Prerequisite to Meaning", Mathematics Teacher, November 1950; "Developing A Logical Concept in Elementary Mathematics", Mathematics Teacher, December, 1955; "Variable Paradox", School Science and Mathematics, June, 1959; "Nonverbal Awareness in the Learning of Mathematics: Critical Unanswered Questions", Cooperative Research Monograph No. 3: Research Problems in Mathematics Education, U. S. Government Printing Office, 1960; "The Case for Basic Research on Theory of Instruction", American Mathematical Monthly, May, 1960; [with Byrl Sims] "The UICSM Teacher Training Films", American Mathematical Monthly, August-September, 1960; "The Role of Learning by Discovery in Manual Skills", Proceedings of the Second Faculty Workshop of the University of Illinois College of Dentistry, Report released October, 1962; Forthcoming: Several brief contributions to a book based on The Interdisciplinary Work Conference on Paralanguage and Kinesics, held at Indiana University, Bloomington, 17-19 May, 1962. "Report of the Committee on Strengthening the Teaching of Mathematics", to appear soon in the American Mathematical Monthly. Principal investigator, psychobiological research on nonverbal learning and communication. Credited in the fifth (forthcoming) edition of Leaders in Science with identification of the nonverbal awareness stage in abstract human learning.

Joseph R. Hoffmann

Born in Los Angeles, California, in 1931. BA [mathematics] in 1952 from Wheaton College; MA [mathematics] from University of Southern California in 1957. Taught at El Camino College, California, 1954-1957; North High School, Torrence, California, 1957-58; Fullerton Junior College, California, 1958-62. NSF Academic Year Institute at Notre Dame, 1959-60. Member of UICSM staff, 1962- .

Allen H. Holmes

Born in Grand Rapids, Michigan in 1937. BA [mathematics] from Carleton College. Graduate work in mathematics at the University of Minnesota. Taught junior and senior high mathematics (UICSM) at the St. Paul Academy, St. Paul, Minnesota, 1959-61. Teacher of mathematics at University of Illinois High School, 1961- .

Edgar B. Horne

Born in Vancouver, B.C., Canada, in 1925. Attended University of British Columbia, 1943-47, 1948-50, 1953-54. BASc [Civil Engineering] 1947, BA [mathematics and physics], 1950, MA [education, mathematics], 1958. Proceeding to PhD in Mathematics Education at University of Illinois [est. 1965]. Teaching experience in British Columbia schools: Nelson Junior High, 1950-53; Templeton Junior High, Vancouver, 1954-57; King Edward High School, 1957-60; Lecturer in Mathematics, 1953-54, and Lecturer in Mathematics Education, 1957, University of British Columbia, Lecturer and Instructor in College of Education, Victoria College, 1960-62. Shell Merit Fellow, Stanford University, 1959. President of B. C. Association of Mathematics Teachers, 1959-60. Member of B. C. A. M. T., Phi Delta Kappa, Canadian Association of University Teachers, NCTM.

Robert L. Johnson

Born in La Junta, Colorado, in 1929. AB [mathematics], Colorado State College, 1951; AM [mathematics] Colorado State College, 1952; MS [teaching mathematics], University of Illinois, 1960; completing requirements for PhD in mathematics education at the University of Illinois [estimate 1964]. Taught at Phoenix Union High School, 1952-55; Central Washington College of Education, Ellensburg, Washington, 1955-59; Colorado State College, 1960-62 (on leave); University of Illinois High School, 1962- . Publication: [co-author] Elements of Set Theory, Allyn and Bacon, 1962, Member of NCTM, MAA, Colorado Mathematics Council.

Robert J. Kansky

Born in Newark, New Jersey, in 1937. BS [education], Florida State University; MA [mathematics] University of Illinois, 1962. Taught at Melbourne [Florida] High School, 1959-61; University of Illinois High School 1962- . Joined UICSM staff as a Research Associate in 1962. Member of Phi Kappa Phi, Omicron Delta Kappa, Pi Mu Epsilon.

Howard Marston

Born in Highland Park, Michigan in 1926. U. S. Navy, 1944-46. BA [mathematics] in 1950 from Wesleyan University; MA [teaching of secondary school mathematics] in 1953 from Columbia University; Chairman of mathematics department of The Principia Upper School since 1959; taught at NSF summer institutes at University of Arizona and University of Illinois since 1958. Publication: "Worktext in Modern Mathematics", Row, Peterson Co., 1962. Member of MAA, NCTM.

William J. Masalski

Born in New Britain, Connecticut, in 1934. BS [mathematics] in 1956 from Central Connecticut State College; MA [education] from Fairfield University [Connecticut] in 1959; graduate work at Wesleyan University [Connecticut], in 1958, University of Maryland in 1959, Yeshiva University in 1960-61, Wayne State University in 1960, Yale University in 1961-62, University of Illinois in 1961 and 1962, and Columbia University in 1961. Taught in the Greenwich, Connecticut, Public Schools 1956-1962. Publication: "A New Tool for Teaching Mathematics: The Overhead Projector", UICSM Newsletter, May 1962. UICSM staff 1962- . Member of NCTM.

M. Eleanor McCoy

Born in Hersman, Illinois, in 1915. AB [history and mathematics] from Carthage College; AM [political science] University of Illinois; summer graduate work at University of Colorado and Central Washington College of Education. Taught mathematics in Fillmore, Illinois, in 1937-39 and in Pekin, Illinois, in 1939-57. Taught in NSF summer institutes at University of Hawaii and University of Illinois. Taught two years of UICSM mathematics in Pekin High School. Joined UICSM staff as associate teacher coordinator in 1957. Assistant professor of education, University of Illinois, 1959- . Publications: "A Secondary School Mathematics Program", The Bulletin of the National Association of Secondary School Principals, May, 1959; "University of Illinois Mathematics Project", Science Education News, December, 1961.

Jane Y. Miyashiro

Born in Honolulu, Hawaii, in 1936. BA [chemistry] in 1958 from the University of Hawaii; MA [teaching] in 1959 from Radcliffe College. Initiating work on PhD in chemistry, 1962- , University of Illinois. Taught at Waianae High and Intermediate School [Hawaii], 1959-60; Instructor [science and education], University of Hawaii, Hilo Campus, 1960-62. Member of Phi Kappa Phi, Phi Beta Kappa, AAAS, Hawaiian Academy of Science, NCTM.

Joseph E. Mueller

Born in Muncie, Indiana, in 1925. BS [education] in 1954 from Butler University; MS [teaching of mathematics] in 1959 from University of Illinois; additional work in Mathematics, University of Illinois in 1959, 1961; Manhattan College, 1961. Taught at Warren Central High School, Indianapolis, Indiana, 1954-58, [Chairman, Mathematics Department] and at Greenwich High School, Greenwich, Connecticut, 1959- [Chairman, Mathematics Department]. NSF Academic Year Institute Fellowship in 1958-59 at University of Illinois. Education assistant at University of Illinois in summer of 1962. Membership in Phi Kappa Phi, Pi Mu Epsilon, Kappa Mu Epsilon, MAA, NCTM.

Josephine McK. Phillips

Born in Montague, New Jersey, in 1914. AB in 1935, and MA in 1941, [mathematics] from Montclair State College; attended Newark College of Engineering, 1942; Ed. D. [teaching of mathematics], Teachers College, Columbia, 1962. Taught in public schools in New Jersey, 1935-48, [interruption for Lt. jg U.S. Coast Guard Women's Reserve, 1943-44]; Longwood College, 1948-51; Montclair State College, 1951-53; Boston University, 1959-62; arithmetic editor, D. C. Heath and Company, 1953-62; Lecturer and consultant in mathematics education, 1960- ; writer on UICSM staff, 1962- . Publications: Articles in The Arithmetic Teacher, School Science and Mathematics, Book Production; fifty filmstrips under the general title "Seeing The Use of Numbers" (Eye Gate House Inc., Jamaica, New York); co-author Learning to Use Arithmetic, Revised Edition, [thirty-two titles] D. C. Heath and Company, Boston.

James H. Rollins

Born in McKinney, Texas, in 1930. BA [mathematics], Agricultural and Mechanical College of Texas, 1952; MS[mathematics education], North Texas State University, 1958; completed the Academic Year Institute for High School Mathematics Teachers, University of Chicago, 1958-59; now completing requirements for PhD in mathematics education at University of Illinois [estimate 1963]. Officer, United States Army, 1952-54. Taught at Ozona [Texas]High School, 1954-57; Grand Prairie [Texas] High School, 1957-58; University of Illinois High School, 1961- . Instructor [mathematics and mathematics education], North Texas State University, 1959-61. UICSM staff 1962- . Member of Phi Delta Kappa, NCTM, MAA.

Walter E. Rucker

Born in Sharon, Kansas, in 1933. Attended University of Wichita, 1956-59; AB [mathematics] in 1959. Now completing requirements for Masters in Mathematics at University of South Carolina. Taught at Fontana, California, 1959-60; Redlands, California, 1960-62. On staff of NSF summer institute at University of Illinois, 1962.

Nancy L. Ruether

Born in Troy, New York, in 1939. Attended State University of New York College at Albany, 1957-61. BS in mathematics [1961]; University of Illinois Graduate College of Education, 1962- . Taught at East Greenbush Central Junior High School, 1961-62. Member of NCTM.

Robert E. Rundus

Born in Blue Rapids, Kansas, in 1928. BA [mathematics], Park College, 1950; EdM, University of Illinois, 1957, further graduate work at University of Illinois. Signal Corps U. S. Army 1950-52; Chief Instructor at Motion Picture and TV Sound Recording School, The Signal School [Ft. Monmouth, N. J.] 1952-54; taught high school mathematics in Saybrook, Illinois, 1954-57; Assistant Professor [mathematics] Park College, Parkeville, Missouri, 1957-58, 1959-60; UICSM teacher assoc. 1958-59; Assistant Professor [education] Northern Illinois University, 1961-62; UICSM staff 1962- .

Walter J. Sanders

Born in Tacoma, Washington, in 1927. BA [mathematics] 1954, and MA [mathematics] 1957, University of Washington; MS [mathematics education], University of Illinois, 1958; additional study in mathematics at the University of Illinois and at Montana State University. Taught at Evergreen High School, 1952-53; North Kitsap High School, 1954-57; Chico State College, 1958-59; Western Washington State College 1959- . Also taught at the Boeing Airplane Company Engineering Orientation School, fall of 1951. Publications: "Interval Graphing" The Mathematics Teacher, April, 1961; "Figure All The Angles" Washington Mathematics, Vol. VII, Issue 1; "Mathematics Programs For The Training of Elementary Teachers" The American Mathematical Monthly, December, 1962.

Robin Steere

Born in Wallasey, Cheshire, England, in 1940. Attended Manchester University, 1959-62. B. Sc. [physics], 1962. Graduate work [education] University of Illinois. On UICSM staff as Research Assistant, 1962- .

Steven Szabo

Born in Detroit, Michigan, in 1929. B. Ed. [mathematics] in 1951 from University of Toledo; MEd in 1960 from Wayne State University; beginning work for PhD in mathematics education at University of Illinois. Taught at Catawba Island Elementary School, Port Clinton, Ohio, 1951-53; Pickett School, Toledo, Ohio, 1955-56; Cass Technical High School, Detroit Michigan, 1956-62; University of Illinois High School, 1962- . Research Associate, UICSM Project, 1962- .

Maurice M. Tatsuoka

Born in 1922, in Japan. Attended Urawa Junior College [natural sciences] 1939-42; SB [theoretical physics] Nagoya University, 1945; MA [psychology] George Peabody College for Teachers, 1951; EdD [educational measurement and statistics] Harvard University, 1956. Research in nuclear physics Nagoya University, 1945-48; Lecturer [freshman algebra], Nagoya Commercial College, 1946-47; Instructor [physics and mathematics], Kinjo Women's College, 1948-49; Junior Research Associate [USAF Research Contract], Educational Research Corporation; Teaching Fellow [education] Harvard University, 1954-55; Research Fellow; Laboratory of Social Relations, Harvard University, 1956-57 and Summer 1958; Assistant Professor [science and mathematics], University of Hawaii, [Hilo Campus], 1956-61; Associate Professor [education], University of Illinois, 1961- . Publications: [with P. J. Rulon, D. V. Tiedeman, and C. R. Langmuir] "The Profile Problem: A Methodological Study of the Interpretation of Multiple Test Scores". Educational Research Corporation, February, 1954; [with D. V. Tiedeman] "Discriminant Analysis", Review of Educational Research, 1954; "The Relationship Between Canonical Correlation and Discriminant Analysis; and A Proposal for Utilization of Qualitative Data in Discriminant Analysis", American Review of Tuberculosis, 1956; "Mathematical Models for Learning Based on Probability Theory", Abstracted in Proc. Hawaiian Academy of Science, 1958; [with F. C. Mosteller] "A Commuting-Operator Model", Studies in Mathematical Learning Theory, [Bush and Estes (eds.)], Stanford University Press, 1959; [with R. Cohn, F. C. Mosteller, and J. W. Pratt]. "Maximizing the Probability that Adjacent Order Statistics of Samples from Several Populations form Overlapping Intervals", Annals of Mathematical Statistics, 1960; [with F. C. Mosteller], "Ultimate Choice between Two Attractive Goals: Predictions from a Model", Psychometrika, 1960; [with R. Lachman, and W. Bonk] "Human Behavior during the Tsunami of May, 1960", Science 133: 1405-1409: [with D. V. Tiedeman], "Statistics as an Aspect of Scientific Method in Research on Teaching", in Handbook of Research on Teaching, [N. L. Gage (ed.)], A. C. E. [in press]; [with J. A. Easley], Introduction to Scientific Thought in the Physical Sciences, [a college level general science text], Wadsworth Publishing Co. [in press].

Clifford W. Tremblay

Born in Passaic, New Jersey, in 1933. BA [education], 1955, George Washington University; MA [mathematics], New Jersey State College at Montclair, 1957. Taught at Hackensack High School [N. J.] 1956-62; taught Summer Workshop at Fairleigh Dickinson University in UICSM First Course, 1960; two workshops in First Course during school year 1960-61 at Ridgewood, N. J., and Hackensack, N. J.; taught NSF Institute, University of Illinois, summer, 1962. Membership in Phi Delta Kappa, NCTM, NEA.

Charles Van Horn

Born in South Dakota in 1920, BS [chemistry] from Nebraska State College, AB [psychology] and MA [psychology], University of Southern California; PhD [psychology] expected from USC in 1963. Research associate in Cinema, USC, 1952-55, in Psychology, USC, 1956-58; engineering psychologist, North American Aviation, 1958-59; research director, UICSM-NETRC Math Film Study, University of Illinois, 1959-61; evaluator UICSM Project 1961- . Present Director Coop Research Project 1566, 1961- . Publications: [with W. Rose] "Theory and Practice of Pre-Production Film Testing", Audio-Visual Review, 1956; [with G. L. Bryan and J. W. Rigney] "An Evaluation of Three Types of Information for Supplementary Knowledge of Results in a Training Technique", U. S. Calif. Elect. Personnel Res. Tech. Rep., No. 19, 1957, [with J. W. Rigney and Beverly Newcomb] "A Study of the Relationships Between True and Relative Motion as Perceptual Tasks in Two Display Contexts", U. S. Calif. Elect. Personnel Res. Tech. Rep., No. 20.

Herbert E. Vaughan

Born in Ogdensburg, New York, in 1911. Attended University of Michigan 1928-35; BS [civil engineering] 1932; BS [engineering mathematics], 1932, PhD [mathematics, 1935]. [Dissertation: On Local Betti Numbers.] Taught at Brown University, 1935-36; A. H. Lloyd Fellow at University of Michigan, 1936-37; University of Illinois, 1937- . Professor of Mathematics at the University of Illinois since 1959. Mathematician on UICSM staff since 1955. Author: [with M. Beberman and UICSM staff] High School Mathematics Units 1-11. Publications: "On Locally Compact Metrisable Spaces", Bulletin of the American Mathematical Society, August, 1937; "On Locally Bicompact Spaces", Fundamenta Mathematicae, 1938; "Well-ordered Subsets and Maximal Members of Ordered Sets", Pacific Journal of Mathematics, September, 1952; "On Two Theorems of Plane Topology", American Mathematical Monthly, August-September, 1953; "Characterization of the Sine and Cosine", American Mathematical Monthly, December, 1955; "On the Irrationality of Roots", American Mathematical Monthly, June-July, 1960. Membership in Phi Beta Kappa, Sigma Xi, AMS, Association for Symbolic Logic, MAA, NCTM, AAUP.

Herbert Wills

Born in Downers Grove, Illinois, in 1930. BS [math education] 1952 and MS [math education] 1955 from Eastern Illinois University. Graduate work at University of Illinois 1958- . Taught mathematics at Kinmundy-Alma [Illinois] High School 1955; York Community High School [Elmhurst, Illinois] 1955-61. Visiting Lecturer at University of New Mexico, Arizona State University, University of Vermont, and University of Illinois, summer 1959; Co-Director NSF Institute New Mexico Highlands University 1960; Instructor 1961 NSF Summer Institute Wayne State University. UICSM Staff 1961- . Publications: "Animated Function Notation", UICSM Newsletter, April, 1962, "Th UICSM Programed Instruction Project", The American Mathematical Monthly, October, 1962. Member of ICTM, NCTM, CASMT, Chicago Men's Mathematics Club.

Robert W. Wirtz

Born in DeKalb, Illinois, in 1913. BS [education], Northern Illinois State Teachers College [now Northern Illinois University], 1933. Taught elementary school in Evanston and Glencoe, Illinois. Graduate work [education] Northwestern University and University of Chicago. Formerly Director of Research and Engineering, and currently staff Research Associate, Vari-Type Corporation. Mathematics consultant for Bucks County [Pennsylvania] Public Schools. Author [with W. W. Sawyer and M. Botel] Math Workshop for Children, grades 1-6, Encyclopedia Britannica Press, 1962. Publication: "Non Verbal Instruction" [will appear] The Arithmetic Teacher, February, 1963.

Martin S. Wolfe

Born in Washington, D. C. in 1927. BS [mathematics], University of Maryland; MEd., University of Maryland; MS [mathematics education], University of Illinois. Taught junior high school mathematics in Washington, D. C. in 1951-57 and University of Illinois High School, 1958- . Taught at NSF Summer Institute, University of Illinois, 1961. Member of NCTM.

Russell Zwoyer

Born in Chicago, Illinois, in 1933. BS [education], Illinois State Normal University, 1960; AM University of Illinois, 1962; further graduate work at University of Illinois. Taught at York Community High School, 1960-61. Attended Academic Year Institute 1961-62. Taught at University of Illinois High School and Mathematics Extension Course in Downers Grove, Illinois, 1962- . Member NCTM, Kappa Mu Epsilon, Chicago Men's Mathematics Club.

Exhibit B

UICSM MATHEMATICS PROJECT

director MAX BEBERMAN
assistant director WILLIAM T. HALE
teacher coordinator GERTRUDE HENDRIX
mathematics editor HERBERT E. VAUGHAN
associate teacher coordinator ELEANOR MC COY

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October 19, 1962

MINUTES

UICSM Advisory Board Meeting
Room 213, Illini Union, University of Illinois, Urbana
October 6, 1962

Board members in attendance:

Prof. E. G. Begle
School of Education
Stanford University
Stanford, California

Dr. Harold B. Cores
Educational Facilities
Laboratory, Inc.
New York 22, N.Y.

Prof. Edwin E. Moise
School of Education
Harvard University
Cambridge 38, Mass.

Dean F. T. Wall
Graduate College
University of Illinois
Urbana, Illinois

Prof. Lee J. Cronbach
College of Education
University of Illinois
Urbana, Illinois

Prof. R. D. James
Department of Mathematics
University of British Columbia
Vancouver, B.C.

Dr. Morris Newman
National Bureau of Standards
Washington 25, D.C.

Dr. John D. Williams
RAND Corporation
1700 Main Street
Santa Monica, California

UICSM staff and guests in attendance:

Max Beberman, William T. Hale;
[afternoon only] John Easley, M. Eleanor McCoy, Alice Hart,
Mahlon Day

The meeting was convened at 9:30 a.m. with opening remarks by Chairman Wall.

Project Director Beberman related the progress of the project since receipt of the NSF grant on April 1, 1962.

(a) Production of programed version of UICSM Unit 1 [18 volumes of approximately 1400 pages] which is now being tried experimentally in nine schools.

(b) Sponsorship of summer institute for 324 teachers.

(c) Recruitment of entire staff as called for in NSF proposal with exception of an applied mathematician and a geometer.

(d) Consultation with Professor Paul Rosenbloom on course content and learning research problems, and proposed consultation with Professor Dick Wick Hall in November on vectorial approaches.

Beberman asked the board for advice in the matter of recruiting a first-rate applied mathematician who would be willing to devote three to five years of practically full-time effort in assuming responsibility for developing application-type content for our secondary curriculum. Several such people had been approached but without success.

It was the opinion of the board that such a quest would be fruitless and that it was probably dangerous to attempt giving one person this kind of responsibility in view of the diversity of application fields which should be considered. The entire board urged that the responsibility for content development be split in the following manner:

(i) ideas to be generated initially by groups of mathematicians to be covered in work sessions [10 days--2 weeks--summers] attended by a subset of the UICSM project staff;

(ii) instructional materials to be developed by the project staff writers under the active supervision of project mathematicians, such as Vaughan.

The work sessions mentioned under (i) would not be text-writing sessions, but would be outlining and brainstorming sessions. The preparation of instructional materials and their classroom trial and evaluation would be the responsibility of the "permanent" project staff.

The board recommended that a short work session [perhaps 10 days] be held prior to the end of the present semester. We would invite applied mathematicians and pure ones to generate ideas on the use of vectors in junior high school mathematics courses. These ideas would then be incorporated into instructional materials during the second semester and tried out at the University High School in 1963-64. Also, this work group would suggest further topics to be considered by a larger group and for a longer period in the summer of 1963. It was further recommended that the summer idea-generating session be repeated annually. Individual members of the summer groups might be invited to spend consulting periods with the staff during the academic year. Such periods might range anywhere in duration from one week to a full year.

[The board suggested that the summer work periods be held in centers with a better climate than Urbana.]

It was suggested that the staff spend two to three years in experimenting with various topics in seventh and eighth grade mathematics before producing a complete seventh grade course. One board member advised that we compare the following two approaches with any given mathematical topic:

(A) initial motivation through applications in such a way that the mathematical content is "abstracted" from the motivating applications

(B) initial motivation to be completely "mathematical"

The board strongly urged that experimentation be carried out on a small scale before a full-year program is decided upon.

[The meeting adjourned for lunch at 12:10 and reconvened at 2:00 p.m.]

Professor John Easley of the UICSM staff described several objective studies which were underway and presented proposals for several more such studies. Several board members suggested that some objective studies might be undertaken which were less closely connected with course content than the ones Easley had proposed. The board agreed that UICSM could make important contributions to mathematics education through educational research despite the fact that intuition appears to be far ahead of science with regard to teaching. "We should like to see science come to the aid of the UICSM, but we would hate to see it become its master."

The formal meeting adjourned at 3:10 p.m., and an informal one convened at 3:40 p.m. at another location. Before adjournment the board tentatively agreed upon a one-day meeting to be held at O'Hare Inn in Chicago, probably on Saturday, March 9, 1963.

Exhibit C

Research and Experimention

From its first years UICSM has depended heavily on reports from classroom teachers using UICSM materials for guidance information for the revision of the materials. Beginning in September 1962, a program of "objective studies" has been initiated to complement teacher-supplied information in the following ways:

1. Provide performance-centered measuring instruments for detecting how well materials written to treat a given topic achieve the objectives associated with that topic; provide "process" information for guidance of revision efforts.

2. Test in an objective manner some of the long-standing guiding hypotheses in accordance with which UICSM materials have always been constructed [e.g., hypotheses concerning delayed verbalization of generaliztions].

3. Gather systematic data concerning the performance of UICSM students as measured against criteria external to the UICSM project [e.g., standardized tests, college grades].

4. Systematically explore promising possibilities for more effective and efficient evaluation procedures.

5. Explore, under controlled, conditions the range of variables which affect mathematics achievement, with the goal of developing new instructional strategies and procedures.

The ultimate purpose of these studies in every case is to complement the reports submitted by teachers in providing information for the guidance of course content revision. Subsidiary purposes are to provide inquirers external to UICSM with more complete, explicit, and rigorous information concerning newly-developed UICSM materials than has hitherto been available, and to contribute to the artistry, the technology, and the scientific knowledge which underlie all production and use of instructional materials.

Fundamental to the planning and conduct of these studies is the conviction that, over the long run, continuing improvement in the effectiveness and efficiency of instruction can take place only in the presence of a consciously and systematically developed body of empirical and theoretical knowledge of the instructional process. The effort to extend and apply this body of knowledge is the unifying principle underlying our broad range of activity in objective studies. The starting points of our work include the wide and deep experience accumulated by staff members over ten years of production and use of new curriculum materials, and the relevant knowledge and technology available from psychology, logic, philosophy of science,

and other disciplines. Our task is to intergrate these and extend the result in the directions required by our work in curriculum development. Our method is that of experiment cum theory which is standard in the experimental sciences generally.

The following is an account of the "objective studies" in progress on November 25, 1962. [Mention is made of two studies to be initiated in September 1963.]

Performance and "Process" Information for Revision of Materials

There are five studies of student performance and instructional processes in progress: two ad hoc studies which are well-advanced, two indefinite-duration activities underway, and one further project planned for initiation in September 1963.

Two objectives studies are occasioned by the self-instruction work mentioned on page 3 of this proposal. The first is aimed at determining how well the performance objectives of the self-instruction books were achieved; the second at providing information concerning the student-material interaction which takes place when self-instruction material is used.

The achievement study is being directed by John Gilpin. Nine tests evaluating performance objectives of Unit 1 self-instruction materials were constructed and sent to the participating schools, as well as to eight "control" schools who agreed to give the tests at the appropriate points in their regular class coverage of UICSM Unit 1. More than half of the completed tests have been returned, and the remainder are coming in regularly. Data reduction is proceeding apace [see page 6 below], and completion of the study is expected in March 1963. Products will include detailed objective data concerning the performance capabilities achieved by the self-instruction students [these data will be supplied to the self-instruction materials revision staff], a set of revised tests which will be distributed to the UICSM Newsletter mailing list, and a statistical discussion of the outcome of the self-instruction experimental design.

In the study of student-material interaction, the worksheets on which the self-instruction students write their answers are being analyzed for information useful to the staff revising the self-instruction materials. More than half of the worksheets have been returned by the participating schools, and reduction and analysis of these data have begun. In addition, a round of visits in person to the experimental and control schools has yielded some valuable insights into the process of self-instruction. Martin Wolfe is heading this particular study and O. Robert Brown, Jr. is handling the administrative arrangements of both.

The two studies just described are designed to yield several general benefits beyond their designated immediate products. First, they have provided the opportunity for the staff to gain extensive experience in

objective-studies work, and to evolve efficient working relationships and procedures. Second, they have provided material for experimentation on improved methods of handling data of the kinds and amounts we shall probably be called upon to deal with in the future (see page 6 below). Third, they are yielding valuable findings or conclusions relevant to future work in self-instruction. Two examples of the latter are (a) indications of several ways in which the mechanics of self-instruction can be made simpler and more effective, and (b) some findings concerning the operation of social context in student interaction with discovery exercises and concerning the ways in which self-instruction materials, in their present form, fail to supply this context and thus have frequently failed to bring students to make the desired discoveries.

An activity relevant to objective studies work of many sorts is the operational definition of selected objectives of mathematics instruction by the accumulation of a classified pool of test items. Items for a pool of this kind are being drawn from tests and quizzes used by UICSM teachers in the past, from the self-instruction tests, and from miscellaneous other sources. The systematic expansion of this pool, under the direction of J. A. Easley, Jr. and Charles Van Horn, will be a continuing activity of the staff, and it is expected that the availability of the pool will facilitate teaching-studies of many sorts. In addition, our understanding of our aims in teaching will increase as we continually evaluate test items for their relevance to those aims.

A further study has just begun in which John Gilpin and O. Robert Brown, Jr., in cooperation with course content improvement writers, are attempting to establish the student-performance objectives toward which the new curriculum shall be directed. It is expected that this cooperative activity will continue indefinitely while the new curriculum is being developed. Expected yields are a more powerful and informative testing program than has been attempted previously, and a detailed description of the content of the curriculum, stated in performance terms. In addition, the explicit character which decisions to modify objectives will take on under this regimen will encourage writers to distinguish between objectives and strategies and thus enable them to produce alternative treatments which differ only in strategies.

A study designed to gather "process" data regarding the operation of the new curriculum in the classroom is scheduled to begin in September 1963. Under this study a systematic program of class observation will be carried out whose primary objective will be the gathering of information for revision and for the improvement of instructional strategies and procedures generally; guidance and development of the individual teachers involved will not be an objective of these observations. A prerequisite study devoted to developing systematic procedures of observation and efficient methods for training observers is already in progress [see page 7].

Experimental Tests of Fundamental Principles of Instruction Long-Accepted by UICSM

Two studies, both well underway, seek to ascertain the validity of hypotheses which have been basic to the preparation of instructional materials by UICSM for many years.

From the earliest days of the project the UICSM staff has had a basic interest in the experience of mathematical discovery. It has operated on the assumption that one of the most important aspects in training mathematics students is to see to it that they experience appropriate instances of discovery. As a consequence of informal experimentation, it has been believed that it is possible to arrange for discovery to occur, and to do so in such a way that discoveries will occur reliably and frequently for essentially all students. Further, we believe that the frequent experience of discovery results in a much higher and more favorable level of motivation than that usually observed in mathematics classrooms. A study, whose objective is to develop sensitive and reliable methods for observing discovery-related events in individual students, so that these hypotheses may be experimentally evaluated was begun by J. A. Easley, Jr. in mid-September 1962, in cooperation with the Coordinated Sciences Laboratory of the University of Illinois.

The strategy of the study is to record physiological variables such as heart rate, galvanic skin response, and parameters of the breathing cycle and look for changes which occur in conjunction with discovery experiences. The increased motivation which seems to accompany the use of well-prepared discovery exercises has been attributed to "intrinsic reinforcement" occuring in concurrence with "discovery". If "intrinsic reinforcement" events do occur, it should be possible to detect and record their physiological correlates.

The Coordinated Sciences Laboratory has provided time on their computer-based automated instruction device called 'PLATO' and has constructed and debugged instruments for recording and processing data on heart rate of students while they work on PLATO. [At a later time we expect to explore the usefulness of respiration and GSR measures in addition to heart rate.] UICSM staff members have written some forty pages of instructional material especially for this study which are designed to induce a variety of discovery experiences in a single session. Subjects have been selected and scheduled and a few have been run through this program. Current attention is being directed to the establishment of baselines necessary for the main program of research. Preliminary results should be available within the next two months.

Another hypothesis of long-term standing in UICSM is that achievement is accentuated if a generalization is first attained preverbally by the student by means of a set of discovery exercises, and only later verbalized by him. In March 1961 a study was undertaken by Martin Wolfe aimed at discovering whether this hypothesis holds true for second-year UICSM students who enter the experimental units with an academic year of experience of learning by discovery.

Two sets of self-instruction materials were prepared, one constructed in discovery fashion, the other in expository fashion; an achievement test, a test of transfer for the subject-matter covered by the self-instruction books, and an attitude questionnaire were constructed. Both the self-instruction materials and the tests were tried out with pilot groups and revised in the light of the data gathered. The revised materials were sent to the cooperating schools [approximately 360 students are participating in the main experiment], and about two thirds of the data have been returned. When the remainder of the data is in hand, the data will be analyzed and the findings reported.

In the analysis of the data attention will be given to transfer and to possible differential effects of the experimental treatments on high and low achievers on previous work. [At a later date a study similar to this one may be undertaken with students naive with respect to learning by discovery.]

Performance of UICSM Students with Respect to External Criteria

Three objective studies in progress are concerned with characteristics of performance of current UICSM students as compared with those of students trained with "conventional" materials. The first of these, under the direction of Maurice Tatsuoka, is analyzing data from Cooperative Achievement Tests collected from 1600 UICSM and 600 control students. Results will provide a gage of the achievement of UICSM students on the objectives specific to "conventional" courses. The data will be analyzed by ability range to detect any differential effects which are functions of the ability variable. The study of algebra achievement is scheduled for completion in early 1963. Future activities will include a similar analysis of UICSM and "conventional" geometry courses.

A second study, directed by Charles Van Horn [supported by a special grant from the Cooperative Research Division of the U.S.O.E.], is concerned with student growth in intellectual abilities as this growth is affected by two different types of training in first year algebra, specifically, UICSM First Course and "conventional" elementary algebra. This study attempts to measure some of the changes which the UICSM courses are designed to produce but which are not measured by tests of the variety commonly used in comparison studies. In particular we expect to find an increased mathematical aptitude after a year of UICSM training.

Subsidiary questions are concerned with the possibility of explicating 'mathematical aptitude' in terms of more elementary intellectual abilities, with possible sex differences in mathematical aptitude and achievement as these may be affected by the two types of training in algebra, and with the kind of measures which will function as predictors of success in algebra courses of the two types. Progress of the study to date includes assembling and editing 30 tests of various intellectual abilities, and the administration of these to pupils in approximately 50 ninth grade algebra classes. Data from these tests are being processed currently, and criterion tests for UICSM Units 1-3 are being prepared for distribution. Near the end of the year the 30 tests will be readministered, and the data from the two administrations will be factor-analyzed. Anticipated completion date for the study is November 30, 1963.

A third study, being conducted by J. A. Easley, Jr. involves the collection of questionnaire data concerning UICSM students and "conventionally" trained students with respect to their behavioral choices and capabilities in the college environment. Questions to be investigated include the following: Do UICSM students fare better than "conventionally" trained students of comparable ability on the CEEB mathematics achievement test? Do UICSM students select college courses in mathematics more frequently than "conventionally" trained students? Are there systematic differences between UICSM and conventional students in mathematics achievement as measured by college grades? Do UICSM students select courses and curricula closely related to mathematics [e.g., physics, engineering] more frequently than "conventionally" trained students. Are there significant sex differences connected with any of these measures? We expect, also, to evolve improved procedures for future follow-up studies as a result of our experience in this study. The questionnaire to be used in the present study is currently being developed. Preliminary testing of the questionnaire, main data gathering, and data processing are scheduled during 1963. Anticipated conclusion date for the study is June 1963.

Evaluation methodology

Four studies are currently in process. Two of these are short-term studies concerned with the mechanics of data reduction, one involves the refinement of procedures and criteria for class-observation as a source of information for revision of course-content improvement materials, and one is concerned with the feasibility of organizing a large part of instructional evaluation in terms of time scores rather than the customary error scores.

The first study, directed by John Gilpin, concerns the application of his new method of data reduction to the data from the nine self-instruction tests, involving some 500 students. The method is designed to handle data reduction tasks which are too large to be handled conveniently by ordinary manual methods, yet too small to justify the employment of automatic methods. Additional advantages are that the resulting organization of the data, a specialized kind of graph called a 'condensed tabulation sheet', makes it possible to see many relationships simply by inspection; and that certain kinds of readout, including 3-, 4-, or n-way complex cross-tabulation of items, are possible which are very cumbersome by either manual or automatic methods. So far, more than half of the tests have been scored, all the steps of the reduction procedure have been tried out and refined to a form satisfactory for the routine progress of work, and a detailed written description of the method has been prepared. The application of the method to the test data is currently proceeding apace.

A closely related study, proceeding under the direction of Martin Wolfe, seeks to compare the efficiency of the data reduction method discussed above with one specfically designed manual method and one specifically designed automatic method. Progress to date includes the correction of the worksheets for the two forms [linear and branching] of the self-instruction book on multiplication of real numbers and the production of a finished condensed

tabulation sheet [experimental reduction method] for the branching form. Reduction of the same data by the other two methods, and production of a condensed tabulation sheet for the linear form are currently in progress. Time and cost figures associated with the three methods are being kept. When the reduction of the data from the book on multiplication has been completed by the three methods, a decision will be made concerning which method will be used for the reduction of the remainder of the worksheet data. In addition, if it seems warranted, an article describing the experimental method of reduction will be prepared for publication in a professional journal.

The observation study, conducted by Eleanor McCoy and Jane Miyashiro, is aimed toward sharpening classroom observation as a tool for revision of experimental instructional materials. It is searching for clearly defined criteria for selecting classroom events which will be useful to course content writers for revising materials, for efficient means for communicating classroom observations to these writers including coding for ready search and retrieval. The formulation of basic skills in observation which should result will be useful in the training of new observers. An observation staff has been selected, schedules set up, and preliminary data reduction methods [dictating machines and McBee Keysort cards] selected. Observation and data collection have been under way for about two months. Preliminary inspection of these data with a view toward a first "metamorphosis" of procedures is presently underway.

The fourth study in evaluation methodology, conducted by John Gilpin, has the following objectives: the refinement of a new general method of instructional research [suitable for use with a wide range of independent variables], using total instructional time as the primary dependent variable; adaptation of this time-oriented research method to the economical development and revision of new curriculum materials; the development of a time-rate-of-progress measure of individual differences in "aptitude" or "learning capacity". The basic strategy of the study involves the use of small numbers of subjects under highly controlled conditions. In order to implement the required large degree of experimental control, a specially outfitted experimental workspace will be prepared and an automatically controlled, general purpose teaching device constructed. This teaching device, based on a multi-speed, remote control motion picture projector [the Perceptoscope], will be sufficiently flexible to accomodate essentially any method of visual stimulus presentation that may be required, will collect data automatically, and will be compatible with added audio and with a variety of operanda. The rationale connected with the use of time as the primary dependent variable in instructional research has been set forth in a general way in a published article. [Gilpin, John. "The Design and Evaluation of Instructional Systems". Audiovisual Communication Review, 10, 2, 1962.] It may be expressed in over-simplified form as follows: if instructional objectives are available in detailed and operational form, and if instructional materials and examinations are designed in such a way as to allow and require the student to recycle until he has met these objectives to some pre-set criterion, the time required for him to complete this process can be used as a measure of his aptitude, of the effectiveness of the materials, and of the difficulty of the subject matter. When the instructional materials and examinations are well constructed for this type of instruction, the time measure will be a sensitive and useful one.

To date the abovementioned article has been prepared, progress has been made on the technical problems of refining and stating instructional objectives in useful form, an experimental space has been procured, and the selection and design of equipment has begun. Currently the work on instructional objectives and on equipment design is continuing. The experimental equipment and space are scheduled to be ready for use in June 1963, at which time the first of a series of exploratory pilot studies will be run using arithmetic drill material and sixth grade subjects. Later work will involve both specially prepared laboratory materials, and selected sequences from actual course content material.

Variables Which Affect Mathematics Achievement; the Development of New Instruction Strategies

Four studies are underway. One of these, under the direction of Easley and Van Horn, is concerned with the relationship between teacher attitude and pupil mathematics achievement, and seeks to answer questions such as the following: How do teacher attitudes toward course content change with accumulation of experience and/or skill? How does pupil achievement in mathematics vary with teacher attitudes toward the topics covered? What is the dimensional structure of teacher attitudes toward the content of mathematics? To date, several hypotheses have been formulated for investigation, a questionnaire is being designed, and arrangements for the distribution of questionnaires have been planned. After the questionnaires have been returned and the data tabulated and analyzed, those scales constituting stable measures will be compared with available criteria of experience and teaching competence. If the results are promising, the scales will be rewritten and the study continued.

Three other studies already mentioned, which are relevant to variables affecting mathematics achievement and the development of new instructional strategies, are those concerned with the differential effects of UICSM and "conventional" training in elementary algebra on student growth in intellectual abilities, the physiological correlates of intrinsic reinforcement, and the use of time as dependent variable in instructional research.

Exhibit D

A Patterened Review of Elementary School Arithmetic

It is our belief that as children "do mathematics", the fundamental ideas of that discipline unfold. The nature of the materials exhibited herein is such that a quick glance through the sample pages reveals little. There is almost no use of verbalized communication. Consequently, the rationale of the development emerges only as the arithmetic is done. We urge the present reader to work through the sample pages inserting answers in the blanks provided.

When such work is done by a student, the forward movement of activities becomes apparent as he proceeds. He experiences many surprises as unexpected patterns emerge. As his curiousity is aroused, he is motivated to continue working. If we have been successful, the student will be rewarded for his extra work when his tentative ideas about the patterns are either validated or replaced in the light of new data. At the same time, the student will observe certain new patterns emerging.

The basic thinking behind this organization of the material comes from a conviction that posing challenging questions--less than obvious questions--is fundamentally sound pedagogy. The student who discovers an unexpected pattern in the work he has been doing has become aware of a mathematical question which might have been very difficult to express directly to him. Similarly, the pattern which suddenly breaks down poses other mathematical questions.

A word about format--Two tracks are provided in the sample [top and bottom]. In addition, a third track has been written. All of these are to be displayed on the same page. Each of the tracks is a continuous development; however, we offer the student a large degree of freedom in his choice of activities. He may work in one track as long as he desires. He may switch to another track at any time if the "puzzles" there seem more intriguing.

Gallons, Quarts, Half-pints — — Gal., Qt., ½Pt. — — g, q, hp — —

You already know that:—
1 gallon = 4 quarts, and
1 quart = 4 half-pints
so, 1 gallon = ____ half-pints

Shorthand:

2 gal. only ------ 2 g, 0 q, 0 hp
2 qt. and 3 half-pints 0 g, 2 q, ___ hp

Rules:

Let's agree to use the smallest numbers we can: so—
5 half-pints will be written:
0 g, 1 q, 1 hp

According to our rule, we will write —

6 quarts as ---------- ___ g, ___ q, ___ hp
7 half-pints as --------- ___ g, ___ q, ___ hp
8 half-pints as --------- ___ g, ___ q, ___ hp
9 half-pints as --------- ___ g, ___ q, ___ hp
5 quarts and 3 half-pints as - ___ g, ___ q, ___ hp
6 quarts and 6 half-pints as - ___ g, ___ q, ___ hp
7 quarts and 10 half-pints as - ___ g, ___ q, ___ hp

Factors and products

Using counting numbers that are greater than 1 — that is, 2, 3, 4, 5, 6, 7, etc., — make as many of the following into true sentences as possible:

3 × 3 = 9
___ × ___ = 35
___ × ___ = 5
___ × ___ = 49
___ × ___ = 17

7 ÷ ___ = ___
21 ÷ ___ = ___
2 ÷ ___ = ___
15 ÷ ___ = ___
25 ÷ ___ = ___

___ × ___ = 4
___ × ___ = 45
___ = 11 ÷ ___
___ = 3 ÷ ___
8 = ___ ÷ ___
13 ÷ ___ = ___
___ = 23 ÷ ___
___ × ___ = 77

___ = 29 ÷ ___
25 = ___ × ___
___ × ___ = 27
31 ÷ ___ = ___
___ = 19 ÷ ___
___ × ___ = 37
37 ÷ ___ = ___
___ × ___ = 39

Express the same amount in two ways:

Not according to our rule	according to our rule
0 g, 5 q, 3 hp;	___ g, ___ q, ___ hp
0 g, 4 q, 10 hp;	___ g, ___ q, ___ hp
1 g, 9 q, 13 hp;	___ g, ___ q, ___ hp
0 g, 7 q, 0 hp;	___ g, ___ q, ___ hp
0 g, 0 q, 9 hp;	___ g, ___ q, ___ hp
0 g, 7 q, 9 hp;	___ g, ___ q, ___ hp

not according to rule	according to rule
0 g, 0 q, 12 hp	--- ___ g, ___ q, ___ hp
0 g, 0 q, 23 hp	--- ___ g, ___ q, ___ hp
0 g, 0 q, 35 hp	--- ___ g, ___ q, ___ hp
0 g, 0 q, 5 hp	--- ___ g, ___ q, ___ hp
0 g, 4 q, 0 hp	--- ___ g, ___ q, ___ hp
0 g, 4 q, 5 hp	--- ___ g, ___ q, ___ hp
0 g, 0 q, ___ hp	--- 1 g, 3 q, 2 hp
0 g, 0 q, ___ hp	--- 1 g, 2 q, 3 hp
0 g, 0 q, ___ hp	--- 3 g, 2 q, 1 hp

Which of the given numbers from the bottom section of the previous page are not the products of counting numbers greater than 1?

___ ___ ___ ___ ___

___ ___ ___ ___ ___

___ ___ ___ ___ ___

Are there other such numbers that are less than 40?

___ ___ ___ ___ ___

List the first 10 even numbers that are also multiples of 3.

___ ___ ___ ___ ___ ___ ___ ___ ___ ___

List the first 10 odd numbers that are also multiples of 5.

___ ___ ___ ___ ___ ___ ___ ___ ___ ___

List the first 10 numbers that are multiples both of 3 and of 4.

___ ___ ___ ___ ___ ___ ___ ___ ___ ___

List the next 10 numbers that should be included in an extension of the list on the left — greater than 40.

___ ___ ___ ___ ___ ___ ___ ___ ___ ___

More about Gal., Qt., & $\frac{1}{2}$ pts... g, q, hp

What do you notice about each group of three examples at the top of page 3?

To save space on this page and page 5, let us agree that:

___, ___, ___ means ___ g, ___ q, ___ hp

And, in the spaces above the blue lines, we will continue to use the smallest number possible.

Also, let us agree that numerals written in parentheses on these two pages refer to the number of half-pints.

'(19)' means 19 half-pints

(19)... 1, 0, 3	( )... 2, 2, 1
(10)... ___, ___, ___	(22)... ___, ___, ___
(19)... ___, ___, ___	( )... 3, 3, 3
(9)... ___, ___, ___	( )... 0, 3, 0
(8)... ___, ___, ___	(21)... ___, ___, ___
( )... 1, 0, 1	(33)... ___, ___, ___
(18)... ___, ___, ___	( )... 1, 1, 2
(30)... ___, ___, ___	( )... 0, 1, 2
( )... 3, 0, 0	(28)... ___, ___, ___

[illegible]

You know that some counting numbers greater than 1 such as 9, 12, 28, etc., are the product of counting numbers greater than 1.

9 = ___ × 3, 12 = 4 × ___, 28 = 7 × ___

These are called: Composite Numbers

All others are called: PRIME NUMBERS

Complete the charts at the bottom of these two pages listing, IN ORDER, the prime numbers across the top and the composite numbers down the side. Then, place a check '✓' to indicate which prime numbers are the factors of each composite number.

×	2	3	5	7	11	13		
4	✓							
6	✓	✓						
8	✓							
9		✓						
10	✓		✓					
12								
14								
15								
16								

(9)... __, __, __	(15)... __, __, __	(15)... __, __, __	(10)... __, __, __
(7)... __, __, __	(6)... __, __, __	(16)... __, __, __	( )... 0, 2, 2
(2)... __, __, __	(9)... __, __, __	( )... 1, 3, 3	(20)... __, __, __
(8)... __, __, __	( )... 2, 0, 3	(16)... __, __, __	(8)... __, __, __
(3)... __, __, __	( )... 1, 0, 1	(4)... __, __, __	(12)... __, __, __
(5)... __, __, __	(18)... __, __, __	( )... 0, 3, 0	(20)... __, __, __
( )... 2, 2, 3	( )... 0, 3, 0	(21)... __, __, __	(16)... __, __, __
( )... 1, 2, 1	(3)... __, __, __	(2)... __, __, __	(1)... __, __, __
(18)... __, __, __	(9)... __, __, __	(19)... __, __, __	(15)... __, __, __

×	2	3	5					
22								
24								

Look at the two charts. For which composite numbers do you find only one check?

__ __ __ __ __ __ __ __

If the charts were extended to include all composite numbers less than 200, what others would have only one check?

__ __ __ __ __ __ __ __

Only one composite number in these two charts has 3 checks. List it. Also, list all other composite numbers less than 100 which, if the charts were extended, would have 3 checks.

__ __ __ __ __ __ __ __

__gals., __qts., __half-pints... __g, __q, __hp... ... __, __, __....

You will not need to use any number larger than 3 on the top section of these two pages.

1, 2, 3	2, 2, 2	1, 1, 1	3, 2, 2	2, 0, 1
+1, 0, 2	+0, 2, 1	+__, __, __	−1, 0, 3	−1, 0, 2
__, 3, 1	__, __, __	2, 0, 0	__, __, __	__, __, __

Initial amount ..... Twice as much	Initial amount ..... Half as much
1, 1, 2 .... __, __, __	0, 1, 0 .... __, __, __
0, 0, 3 .... __, __, __	1, 0, 0 .... __, __, __
0, 3, 0 .... __, __, __	0, 3, 0 .... __, __, __
0, 2, 2 .... __, __, __	2, 1, 0 .... __, __, __
0, 3, 3 .... __, __, __	2, 3, 2 .... __, __, __

1, 0, 2
× 2
__, __, __

0, 2, 3
× 3
__, __, __

Express each number listed in the left-hand column by an indicated product of the largest number of factors other than 1.

Number	as largest number of factors	no. of factors	no. of different factors
8	2 × 2 × __	3	1
12			
30			
27			
15			
25			
60			

List the numbers less than 150 that can be expressed by an indicated product of exactly 3 different factors from the following group of factors:

2, 3, 5, 7, 11, 13, 17, 19, 23.

30 = 2 × 3 × 5	__ = __ × __ × __
42 = __ × __ × __	__ = __ × __ × __
__ = __ × __ × __	__ = __ × __ × __
__ = __ × __ × __	__ = __ × __ × __
__ = __ × __ × __	__ = __ × __ × __
__ = __ × __ × __	__ = __ × __ × __

A Initial amount	B Twice as much (as A)	C Add 1 qt. and 2 hp. to amount listed in column B.	D Half as much as amount listed in column C.	E From amount listed in column D, take away the amount in column A.
0, 2, 0	1, 0, 0	___, ___, ___	___, ___, ___	___, ___, ___
0, 1, 1	___, ___, ___	___, ___, ___	___, ___, ___	___, ___, ___
1, 0, 0	___, ___, ___	___, ___, ___	___, ___, ___	___, ___, ___
1, 1, 1	___, ___, ___	___, ___, ___	___, ___, ___	___, ___, ___
1, 2, 0	___, ___, ___	___, ___, ___	___, ___, ___	___, ___, ___
___, ___, ___	2, 3, 0	___, ___, ___	___, ___, ___	___, ___, ___
___, ___, ___	___, ___, ___	2, 1, 0	___, ___, ___	___, ___, ___

First — find the smallest number greater than 50 that can be expressed by an indicated product of

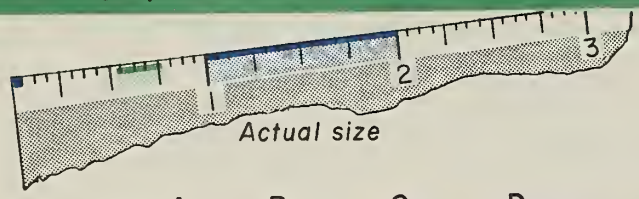
(a) the <u>total number</u> of prime factors as indicated, and

(b) the number of <u>different prime</u> factors indicated.

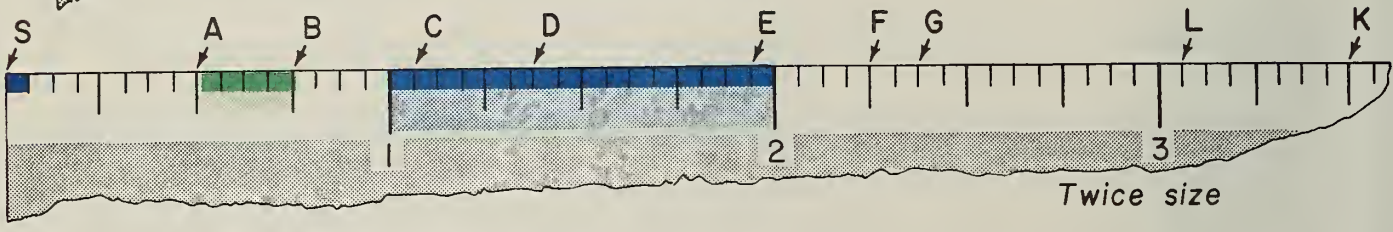
Second — find the largest number less than 100 that can be expressed according to the same pattern of factors.

smallest number more than 50		total number of factors	number of different factors		largest number less than 100
60	___×___×___×___	4	3	___×___×___×___	90
	___×___×___	3	3	___×___×___	
	___×___	2	2	___×___	
	___×___×___×___×___×___	6	1	___×___×___×___×___×___	
	___×___×___	3	2	___×___×___	
	___×___×___×___	4	2	___×___×___×___	

Inches, quarters of inches, and sixteenths of inches..... ___i. ___q, ___s



Because sixteenths of inches are so small, the piece of ruler has been drawn twice size.



$\overrightarrow{SA}$... no inches, 2 quarters, and no sixteenths

To save space, we will leave out the letters.

$\overrightarrow{SA}$... 0 i. 2 q, 0 s	$\overrightarrow{BC}$... 0 i. 1 q, ___s	$\overrightarrow{AD}$... ___.___,___
$\overrightarrow{AC}$... 0 i. 2 q, 1 s	$\overrightarrow{CD}$... ___i. ___q, ___s	$\overrightarrow{DE}$... ___.___,___
$\overrightarrow{SC}$... 1 i. 0 q, 1 s	$\overrightarrow{BD}$... ___i. ___q, ___s	$\overrightarrow{AE}$... ___.___,___

	I	II	III	IV	V	VI	VII	VIII	IX	X	Pattern Indicator
Consider a pair of numbers	4		7	8							a
If not the same, the larger is listed above the smaller	3	5									b
The sum of the pair	7	10	9		13	12	14				a + b
The difference of the pair	1	0				10					a − b
The product of the pair	12			48	36						ab
Twice the sum of the pair	14								30		2(a + b)
Twice the product of the pair	24						98	108		140	2ab
	17							25			a+b+10
	17								105		2ab − 7
	2	0	4	4	10	20	0	6		6	

$\overrightarrow{DG}$... ___•___, ___	$\overrightarrow{CG}$... ___•___, ___	2 $\overrightarrow{AB}$....2× ___•___, ___... ___•___, ___
$\overrightarrow{GK}$... ___•___, ___	$\overrightarrow{GL}$... ___•___, ___	2 $\overrightarrow{BC}$....2× ___•___, ___... ___•___, ___
$\overrightarrow{DK}$... ___•___, ___	$\overrightarrow{CL}$... ___•___, ___	2 $\overrightarrow{AC}$....2× ___•___, ___... ___•___, ___
$\overrightarrow{AE}$... ___•___, ___	$\overrightarrow{SG}$... ___•___, ___	2 $\overrightarrow{SD}$....2× ___•___, ___... ___•___, ___
$\overrightarrow{EG}$... ___•___, ___	$\overrightarrow{GK}$... ___•___, ___	2 $\overrightarrow{DE}$....2× ___•___, ___... ___•___, ___
$\overrightarrow{AG}$... ___•___, ___	$\overrightarrow{SK}$... ___•___, ___	2 $\overrightarrow{SE}$....2× ___•___, ___... ___•___, ___
$\overrightarrow{DF}$... ___•___, ___	$\overrightarrow{SE}$... ___•___, ___	3 $\overrightarrow{AC}$....3× ___•___, ___... ___•___, ___
$\overrightarrow{FG}$... ___•___, ___	$\overrightarrow{DE}$... ___•___, ___	3 $\overrightarrow{CD}$....3× ___•___, ___... ___•___, ___
$\overrightarrow{DG}$... ___•___, ___	$\overrightarrow{SD}$... ___•___, ___	3 $\overrightarrow{AD}$....3× ___•___, ___... ___•___, ___

How many composite odd numbers are less than 100? Express each by an indicated product of factors — which need not be prime.

9 = 3 × 3	___ = ___ × ___	___ = ___ × ___
15 = 5 × 3	___ = ___ × ___	___ = ___ × ___
___ = ___ × ___	___ = ___ × ___	___ = ___ × ___
___ = ___ × ___	___ = ___ × ___	___ = ___ × ___
___ = ___ × ___	___ = ___ × ___	___ = ___ × ___
___ = ___ × ___	___ = ___ × ___	___ = ___ × ___
___ = ___ × ___	___ = ___ × ___	___ = ___ × ___
___ = ___ × ___	___ = ___ × ___	___ = ___ × ___

How many in the list are not multiples of 5? _____

How many are not multiples of either 3 or of 5? _____

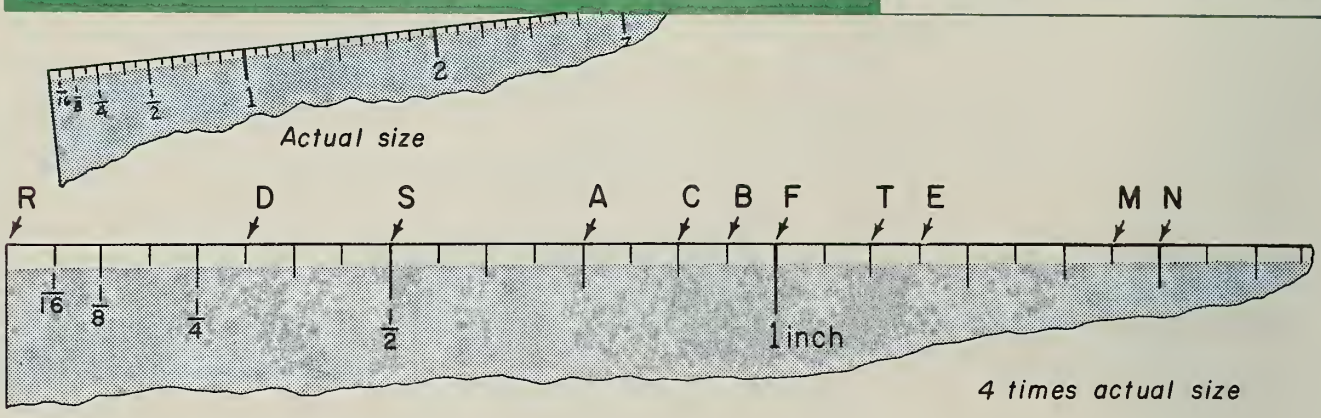
How many are not multiples of either 3 or 5 or 7? _____

List each number, odd or even, less than 100 which is a multiple of 3 as well as a multiple of 4.

___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___

Inches, halves of inches, quarters, eighths, sixteenths.... __ i. __ h, __ q, __ e, __ s



$\overrightarrow{RB}$ is 0 inches, 1 half in., 1 quarter in., 1 eighth in., and 1 sixteenth.

In our shorthand, we could write:

0 i. 1 h, 1 q, 1 e, 1 s

Or, remembering the letters:

$\overrightarrow{RD}$... 0 . 0, 1, 0, 1

$\overrightarrow{SE}$... 0 . 1, __, __, __ $\overrightarrow{DF}$... __ . __, __, __, __ $\overrightarrow{AE}$... __ . __, __, __, __

Complete the following to make true sentences:

Rule. The number listed above the first (or left) blue line must be less than the number listed above the second blue line.

Remember. Indicated multiplications are to be completed before indicated additions unless otherwise specified.

13 = __ + 3 × 4	38 = 3 + __ × 7		
16 = 1 + __ × 5	17 = __ + 5 × __		
29 = __ + 3 × __	__ = 0 + 5 × 11		
17 = 2 + 5 × 3	__ = 2 + 6 × 2	10 = __ + 7 × __	47 = __ + 6 × __
11 = __ + 5 × 2	28 = __ + 6 × 4	25 = 4 + __ × 3	14 = __ + 6 × __
28 = __ + 5 × __	42 = 0 + __ × 7	35 = __ + 7 × __	61 = __ + 6 × __
39 = 4 + __ × 7	70 = 4 + 6 × __	17 = __ + 2 × 8	11 = __ + 4 × __
23 = __ + 5 × __	15 = 3 + __ × 3	23 = 1 + __ × 11	33 = __ + 4 × __
__ = 2 + 5 × 5	17 = 1 + 4 × __	40 = __ + 2 × __	44 = __ + 4 × __
5 = 0 + 5 × __	32 = 0 + 4 × __	__ = 1 + 2 × 31	77 = __ + 4 × __

1 eighth of an inch = ___ sixteenths of an inch.
1 quarter of an inch = ___ eighths of an inch.
1 half of an inch = ___ quarters of an inch.
1 whole inch = ___ halves of an inch.

	inches	halves	quarters	eighths	sixteenths
$\overrightarrow{RS}$...	0	1	0	0	0
$\overrightarrow{SA}$...	___	___	___	___	___
$\overrightarrow{AC}$...	___	___	___	___	___
$\overrightarrow{CB}$...	___	___	___	___	___
$\overrightarrow{BF}$...	___	___	___	___	___
$\overrightarrow{RF}$...	___	___	___	___	___
$\overrightarrow{FM}$...	___	___	___	___	___
$\overrightarrow{RM}$...	___	___	___	___	___
$\overrightarrow{CF}$...	___	___	___	___	___
$\overrightarrow{ET}$...	___	___	___	___	___
$\overrightarrow{CT}$...	___	___	___	___	___
$\overrightarrow{SF}$...	___	___	___	___	___
$\overrightarrow{FN}$...	___	___	___	___	___
$\overrightarrow{SN}$...	___	___	___	___	___
$\overrightarrow{DS}$...	___	___	___	___	___
$\overrightarrow{DN}$...	___	___	___	___	___
$\overrightarrow{AB}$...	___	___	___	___	___
$\overrightarrow{BF}$...	___	___	___	___	___
$\overrightarrow{AF}$...	___	___	___	___	___
$\overrightarrow{FM}$...	___	___	___	___	___
$\overrightarrow{AM}$...	___	___	___	___	___
$\overrightarrow{MN}$...	___	___	___	___	___
$\overrightarrow{AN}$...	___	___	___	___	___
$\overrightarrow{RB}$...	___	___	___	___	___

18 = ___ + 5 × ___

To fill in the space indicated by the green arrow, you might say to yourself:

"What is the remainder when 18 is divided by 5?"

To answer that question, you need to know that 5 × 3 = 15 and that 18 − 15 = 3.

What is the remainder when 23 is divided by 5? _____

What is the remainder when 4 is divided by 7? _____
(4 = ___ + 7 × 0)

15 leaves a remainder of ___ when divided by 7

11 leaves a remainder of ___ when divided by 7

26 leaves a remainder of ___ when divided by 7

To save space, we agree to ask these same questions in shorthand.

15 r ___ d 7	30 r ___ d 9	52 r ___ d10
11 r ___ d 7	40 r ___ d 9	38 r ___ d10
26 r ___ d 7	70 r ___ d 9	90 r ___ d10
20 r ___ d 3	48 r ___ d 8	67 r ___ d10
13 r ___ d 3	20 r ___ d 8	34 r ___ d10
7 r ___ d 3	68 r ___ d 8	101 r ___ d10

A Genetic Development of Number Systems for the Junior High School

The intention of the development is to describe, on an appropriate level of precision, that construction of the real number system which, starting with the finite cardinals, proceeds through the introduction of ratio numbers [equivalence classes of ordered pairs of finite cardinals] to that of measure numbers [lower segments of the ordered set of ratio numbers], and finally, construes real numbers to be relations over the set of measure numbers [i.e. equivalence classes of ordered pairs of measure numbers].

We start with the finite cardinals, numbers which answer the question "How many?" Our treatment of these numbers includes: order, addition, multiplication, symmetric difference, factors, primes, basic principles [commutativity, etc.], place-value notations with different radices, combinations, permutations, and some problems from number theory. The language will include set-terminology, free variables, and quantifiers.

Preliminary to the definition of ratio numbers we discuss measurements, done as classroom experiments, using arbitrary units of length which are exemplified by strips of paper. A measurement is carried out by cutting a "strip-rule" into congruent pieces and counting the number of times such a piece is laid down, end-to-end. The result of such a measurement is an ordered pair (n, m) of cardinals, n being the number of congruent pieces into which the strip-rule was cut, and m the number of times one such piece was laid down. In measuring the length of, for example, a desk-top, one concerns oneself with those measurements in which the piece being laid down never extends beyond an edge of the desk-top. The resulting ordered pairs give information about the strip-measure of the length. For example, a result (3, 8) tells that one of 3 congruent pieces into which a strip-rule is cut can be laid down [at least] 8 times within the limits of the object whose length is being investigated. It is easily seen that the ordered pairs (6, 16), (9, 24) and, in general, each ordered pair (n, m) such that $n \neq 0$ and $8n = 3m$ gives the same information concerning the length of the object. More

generally, if $(n_1, m_1) \sim (n_2, m_2)$ if and only if $m_1 n_2 = m_2 n_1$ then $\sim$ is an equivalence relation among those ordered pairs (n, m) for which $n \neq 0$. The corresponding equivalence classes are, by definition, <u>the ratio numbers</u>. Each member of a ratio number suggests a name for the number. For example, '3/8', '6/16', '9/24', etc. are names for the ratio number which consists of all ordered pairs (n, m), with $n \neq 0$, such that $8n = 3m$. Order, sum, and symmetric difference of ratio numbers can be motivated by obvious considerations of measurement process. Multiplication and division arise, somewhat less obviously, out of attempts to compare the results of measuring the same object with different [but commensurable] units.

The procedure described above associates with each length and each unit of length a set of ratio numbers--the set of all those ratio numbers which are "less than" the measure of the given length with respect to the given unit. Such a set of lower approximations is, of course, a lower segment of the ordered set of ratio numbers. The expectation that the measure of a length with respect to a unit can be given as a single ratio number--in more sophisticated terms, that each lower segment which has an upper bound has a least upper bound--is shown to be over-optimistic by showing that a diagonal of a square is not commensurable with a side. So, the only adequate uniform way of describing a measure is by giving the appropriate lower segment of ratio numbers. It is sufficient to restrict oneself to those lower segments of the set of ratio numbers which have upper bounds but do not have greatest members. By definition, these segments are <u>the measure numbers</u>. Order, sum, symmetric difference, product, and quotient of measure numbers are defined in the usual ways.

The real numbers are introduced as relations among measure numbers in essentially the same way as [but more explicitly than] is done in the existing UICSM Unit I ["The Arithmetic of the Real Numbers"].

It is expected that the motivation furnished, as is indicated above, by the process of measurement will make this rather sophisticated construction appear a natural one to students.

A Senior High School Course in Vector Geometry

We assume that there will be a good deal of intuitive plane and solid geometry in the junior high school, and even some informal deductive geometry. In the senior high school [grade 10 or grade 11], the knowledge so gained of the geometry of E_3 [3-dimensional Euclidean space] will be organized along vectorial lines. [Vector concepts will be developed in the earlier grades from both geometric (invariance of patterns under translations, etc.) and algebraic (vector approach to addition of real numbers, etc.) considerations.] Although not all details of the vector geometry course have been worked out, enough has been done to make it seem likely that it is feasible, on the basis which can be provided by the junior high school years of the program. The remarks which follow are intended to suggest the kind of structure which the writers have in mind.

We consider the space E_3 of points [A, B, C, ...] structured by the 3-dimensional real inner product space T_3 of translations [$\vec{a}$, $\vec{b}$, $\vec{c}$, ...]. In the actual development, T_3 appears first as an abelian group operating on E_3, then as a vector space over the real numbers [a, b, c, ...] and, finally, as an inner product space. As is customary in UICSM texts, these properties of E_3 are "discovered" as formulations of previously noted facts--in this case, facts concerning the geometry of E_3. Translations of E_3 are functions--more specifically, they are mappings of E_3 on itself. [The identity mapping $\vec{o}$ is included among the translations.] If '+' is used to denote application of a function to an agrument, eight of the important properties of translations can be stated as follows:

(i) $A + \vec{o} = A$

(ii) $\exists_{\vec{x}}\, A + \vec{x} = B$

(iii) $\vec{a} \neq \vec{b} \Rightarrow A + \vec{a} \neq A + \vec{b}$

Of these, (i) charcterizes the identity mapping and (ii) and (iii) justify speaking of the translation which maps the point A onto the point B.

(iv) $A \neq B \Rightarrow A + \vec{a} \neq B + \vec{a}$

This last expresses the fact that each translation $\vec{a}$ has an inverse--say $-\vec{a}$--and, noting that this is, itself, a translation, we have:

$$\text{(v)} \qquad (A + \vec{a}) + -\vec{a} = A$$

That the result of two translations is independent of the order in which they are applied--briefly, that E_3 is flat--is expressed by:

$$\text{(vi)} \qquad (A + \vec{a}) + \vec{b} = (A + \vec{b}) + \vec{a}$$

Noting that the resultant of two translations is a translation, and some feeling for the utility of algebraic notations, suggest a second use for '+' which is formulated in:

$$\text{(vii)} \qquad A + (\vec{a} + \vec{b}) = (A + \vec{a}) + \vec{b}$$

[This second use--essentially to denote composition of functions--is exemplified by the second '+' on the right side of (vii).] It also appears convenient to take account of the content of (ii) and (iii) by introducing subtraction of points by:

$$\text{(viii)} \qquad B - A = \vec{a} \iff A + \vec{a} = B$$

These eight statements are [as is easily proved] equivalent to four, which we adopt as postulates:

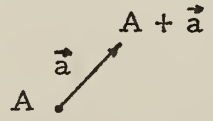
(0_1) Translations form an abelian group

(1) $(A + \vec{a}) - A = \vec{a}$

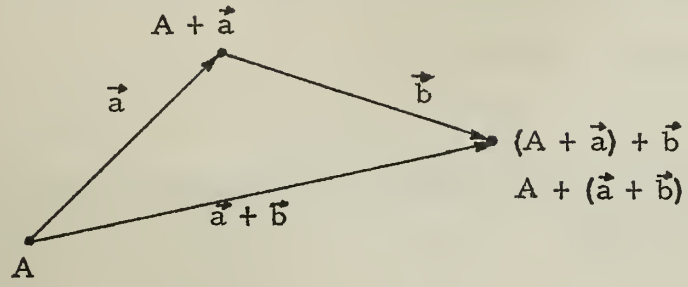
(2) $A + (B - A) = B$

(3) $(B - A) + (C - B) = C - A$

Of the last three, (1) and (2) are equivalent to (viii) and, by virtue of (1) and (2), (3) is equivalent to (vii). Also, (1) and (2) justify the use of arrows, in pictures of E_3, to represent translations:



and (3) justifies picturing the resultant of two translations by the usual diagram:



Besides the postulates, we have two definitions:

$$\vec{b} - \vec{a} = \vec{b} + -\vec{a}, \qquad A - \vec{a} = A + -\vec{a}$$

On this basis we can derive a number of theorems--for examples:

$$(A - B) - (C - D) = (A - C) - (B - D)$$
$$(A + \vec{a}) - (B + \vec{b}) = (A - B) + (\vec{a} - \vec{b})$$
$$A + (B - C) = B + (A - C)$$

which, in view of the interpretation of the algebra, state results concerning constructions based on parallelograms. One can also show [using a corollary of the first of the theorems given above] that, given a point O, there is a canonical correspondence between points and translations, given by the two mappings:

$$P_O(\vec{a}) = O + \vec{a}, \qquad T_O(A) = A - O$$

*

It is of interest, as testifying to the algebraic simplicity of the system, that it can also be based on the following six postulates which concern only the three kinds of subtraction:

$$\vec{a} - (\vec{b} - \vec{c}) = \vec{c} - (\vec{b} - \vec{a}) \qquad \vec{a} - (\vec{a} - \vec{b}) = \vec{b}$$
$$A - (B - \vec{c}) = \vec{c} - (B - A) \qquad A - (A - \vec{b}) = \vec{b}$$
$$A - (B - C) = C - (B - A) \qquad A - (A - B) = B$$

augmented by the three definitions:

$$-\vec{a} = (\vec{a} - \vec{a}) - \vec{a}, \qquad \vec{a} + \vec{b} = \vec{a} - -\vec{b}, \qquad A + \vec{a} = A - -\vec{a}$$

[In proving that these imply our previously chosen postulates, an essential lemma is '$A - A = \vec{b} - \vec{b}$'.]

*

The next step in developing postulates for geometry is to introduction multiplication of a translation by a real number. With the obvious interpretation of this operation, one has:

$$\text{(ix)} \qquad (A + \vec{a}a) + \vec{a}b = A + \vec{a}(a + b)$$
$$\text{(x)} \qquad (A + \vec{a}a) + \vec{b}a = A + (\vec{a} + \vec{b})a$$
$$\text{(xi)} \qquad (\vec{a}a)b = \vec{a}(ab)$$
$$\text{(xii)} \qquad \vec{a}1 = \vec{a}$$

from which one easily derives the statements more usually adjoined to postulates for an abelian group to obtain postulates for a vector space. We adopt these, replacing (0_1) by:

(0_2) Translations form a vector space [over the real numbers].

We are now justified in calling translations <u>vectors</u>. After introducing [and discussing] the notions of a linearly dependent [or independent] sequence of vectors, we can make the usual elementary applications of vectors to the proof of such theorems as the one concerning the concurrency of the medians of a triangle. [This can be done before the notions of line and triangles are introduced on the basis of our postulates, or later, whichever order pedagogical needs dictate.] To deal with lines and planes we introduce two relations by two postulates:

(4) A, B, and C are collinear $\Longleftrightarrow$ (B - A, C - A) is linearly dependent

(5) A, B, C, and D are coplanar $\Longleftrightarrow$ (B - A, C - A, D - A) is linearly dependent

A line is a set ℓ such that

(i) ℓ is nondegenerate, and

(ii) $(\{A, B\} \subseteq \ell$ and $A \neq B) \Longrightarrow [C \in \ell \Longleftrightarrow$ A, B, and C are collinear].

A plane is a set p such that

(i) p contains three noncollinear points, and

(ii) $(\{A, B, C\} \subseteq p$ and A, B, and C are noncollinear)
$\Longrightarrow [D \in p \Longleftrightarrow$ A, B, C, and D are coplanar].

We can now define:

$$\overleftrightarrow{AB} = \{X: \exists_x X = A + (B - A)x\}$$

$$\overleftrightarrow{A\vec{a}} = \{X: \exists_x X = A + \vec{a}x\}$$

and show that, if $A \neq B$ and $\vec{a} \neq \vec{o}$, these are lines, the line through A and B and the line through A in the direction of $\vec{a}$, respectively. Similar procedures apply to planes. Parallelism of lines and of planes can be defined as sameness of direction, and the usual parallel postulate for coplanar lines becomes a theorem. The introduction of notions of [closed] segment and ray and [open] interval and half-line, and of the corresponding planar notions is of course, simple. Separation of planes by lines and of lines by points becomes trivial.

In order to prove that disjoint planes are parallel, that each plane separates space, etc., we need another postulate:

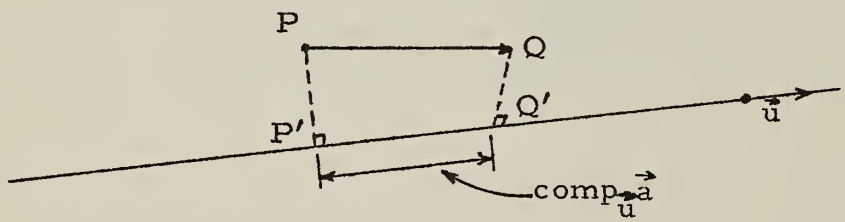
There are 3, but not 4, linearly independent translations.

We incorporate this into our system by replacing (0_2) by:

(0_3) Translations form a 3-dimensional vector space.

Up to now we may have been working in a vacuum; now we are sure that E_3 is 3-dimensional, and we have all that is needed to develop, as far as may seem desirable, 3-dimensional affine geometry. In doing so, one may, of course, choose basis vectors and, in term of the canonical correspondence between vectors and points which obtains, once an origin is chosen in E_3, introduce coordinates in E_3 and deal with lines and planes in terms of equations ["solid analytic geometry"].

The introduction of metric concepts amounts to introducing an inner product in the vector space T. Since translations can be represented by directed segments in E_3, the notion of an inner product can be approached via that of the component of a directed segment with respect to an oriented line--that is, the component of a vector $\vec{a}$



with respect to a unit vector $\vec{u}$. This notion, like the previous notions which led to our earlier postulates, depends on intuitive concepts, here, of perpendicularity and length. Once axiomatized, it will be used to define these concepts.

Suppose that P′ and Q′ are the feet of the perpendiculars from P and Q to a line ℓ in the direction of the unit vector $\vec{u}$. It follows that there is a real number x such that the vector $Q' - P' = \vec{u}x$. It is this number x which is the component of $\vec{a}$ with respect to $\vec{u}$. The fact that "the component of a sum is the sum of the components" [distributivity] suggests that taking components might be considered a kind of multiplication'

$$\text{comp}_{\vec{u}}\,\vec{a} = \vec{a}\cdot\vec{u}$$

The fact that it follows from this that $(\vec{a}a)\cdot\vec{u} = (\vec{a}\cdot\vec{u})a$ suggests defining $\vec{a}\cdot(\vec{u}a)$ to be $(\vec{a}\cdot\vec{u})a$ and, so, forming the inner product of any two vectors. Since it is clear that, for unit vectors, $\vec{v}\cdot\vec{u} = \vec{u}\cdot\vec{v}$, it follows that inner multiplication is commutative. Geometric considerations also show that, for $\vec{a} \neq \vec{o}$, $\vec{a}\cdot\vec{a} > 0$. So, we adopt, as new postulates:

$$(\vec{a} + \vec{b})\cdot\vec{c} = \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c}$$
$$(\vec{a}a)\cdot\vec{b} = (\vec{a}\cdot\vec{b})a = \vec{a}\cdot(\vec{b}a)$$
$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$$
$$\vec{a} \neq \vec{o} \Rightarrow \vec{a}\cdot\vec{a} > 0$$

That is, we replace (0_3) by:

(0) Translations form a 3-dimensional real inner product space.

We now have a basis for the development of Euclidean geometry. In T_3 we can introduce a norm and a notion of orthogonality in terms of which we define distance and perpendicularity in E_3. We can introduce orthonormal bases in T_3 and rectangular cartesian coordinate systems in E_3. We can establish the Cauchy-Schwarz inequality:

$$\vec{a}^2\vec{b}^2 \geq (\vec{a}\cdot\vec{b})^2$$

and define [for $\vec{a} \neq \vec{o} \neq \vec{b}$]:

$$\cos(\vec{a}, \vec{b}) = (\vec{a}\cdot\vec{b})/(|\vec{a}|\,|\vec{b}|)$$

Defining an orthogonal transformation of T_3 to be one which preserves inner products, it follows that $\cos(\vec{a}, \vec{b}) = \cos(\vec{c}, \vec{d})$ if $(\vec{a}, \vec{b})$ and $(\vec{c}, \vec{d})$ are orthogonally equivalent.

It is easy to show that a transformation U of T_3 is orthogonal if and only if it satisfies:

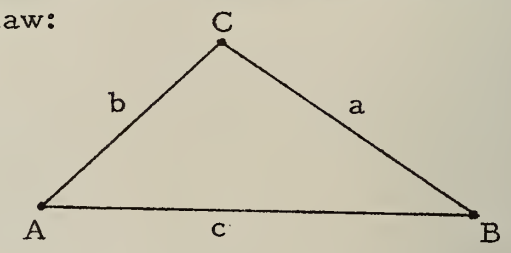
$$(U\vec{a} - U\vec{b})^2 = (\vec{a} - \vec{b})^2, \qquad U\vec{o} = \vec{o}$$

and this suggests the treatment of isometries of E_3 which have a fixed point by means of orthogonal transformations of T_3. There is a 1-1 correspondence between the isometries which leave O fixed and the orthogonal transformations of T which is given by:

$$U_O(A) = O + U(A - O), \qquad U\vec{a} = U_O(O + \vec{a}) - O$$

From the easily established theorem that orthogonal transformations [as defined above] are linear transformation of T_3 onto itself, it follows that isometries are 1-1 mappings of E_3 on itself which preserve collinearity, coplansity, parallelism, perpendicularity, etc. In order to classify isometries we classify orthogonal transformations. [To do the latter, we may find it best to assume, rather than "prove", that each orthogonal mapping preserves at least one direction.]

An angle of E_3 will be defined to be an ordered pair of rays with a common vertex. Its consine is, by definition, the cosine of a corresponding ordered pair of vectors, and the cosine law:



$$c^2 = a^2 + b^2 - 2ab \cos\angle C$$

follows from the vector identities:

$$(\vec{a} - \vec{b})^2 = \vec{a}^2 + \vec{b}^2 - 2\vec{a}\cdot\vec{b} \quad , \quad \vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}| \cos(\vec{a}, \vec{b})$$

At this point one can talk of the "cosine-measure" of an angle [and prove that angles are congruent if and only if they have the same cosine-measure]. The introduction of radian-measure and degree-measure requires the winding-function approach utilized in the UICSM course for the introduction of the circular functions.

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The preceding remarks should suffice to give an idea of the nature of the proposed senior high school geometry course.

EXHIBIT E

PROGRAMED INSTRUCTION PROJECT

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THE UICSM PROGRAMMED INSTRUCTION PROJECT

HERBERT WILLS, Programer, UICSM, University of Illinois

In April 1961 the U. S. Office of Education issued a grant to Professors Max Beberman (UICSM) and Lawrence Stolurow (Dept. of Psychology) of the University of Illinois in support of a study to compare principles of programming mathematics. The content used for this purpose is that contained in the existing experimental textbooks produced by UICSM. Since September 1961 over 200 students have studied from the programmed texts prepared by the Programed Instruction Project (PIP). The information received from this trial of the materials will be helpful in revising them for use next year. In fact, some improvements have already been implemented in later preliminary programs as a result of experience gained from the use of the earlier ones.

In the course of our work during 1961–62, we have developed certain points of view and guidelines for programming mathematics. One of these is concerned with the selection of programming personnel. Recently, considerable discussion has been devoted to determining the relative roles of a "subject-matter specialist" and a "learning-theory specialist" in programming. These discussions have apparently overlooked a third contributor. This individual has the experience and ability to put correct mathematical ideas across to children, and this is the kind of job by which he earns his living. I am referring, of course, to the creative classroom teacher. This person has had much experience in removing students' difficulties, in preparing motivating lessons, in observing individual differences in action, in acquiring a sensitivity to what is easy and what is difficult for children, in asking those questions which enable a student to "see for himself," in pointing to the proper mark on a chalkboard in order to clear up a student's misunderstanding, and in knowing when most students have acquired the desired concept or skill. The creative classroom teacher has been working miracles for years in the face of such discouragements as crowded classrooms, teaching overloads, and a host of nonacademic duties. What could the best of these teachers do if all of their time were devoted to just the one duty of preparing an elaborate lesson [that is, a program] on a single topic?

The creative teacher who has this opportunity faces new problems. Not the least of these is the fact that all of his teaching must be done on paper. In trying to write a program, he soon discovers how much he had relied upon inflections of voice and body movements in "live" teaching. In the UICSM programming work, we try to translate some of these paralinguistic and kinesic components of communication through the medium of type-face changes and layout of material on the printed page. The judicious use of arrows may do something of what finger-pointing does in the live classroom. It may very well turn out to be the case that the creative programmer will need to consult with the commercial artist as well as with the subject-matter and learning-theory experts. A second example of a new teaching-on-paper problem for programmers is that of making sure that students learn correct pronunciations of new terms.

The creative classroom teacher knows how important it is to awaken the student's interest and maintain it. So, the creative programmer will devote considerable effort toward making his presentation interesting and exciting. There is no doubt that immediate knowledge of results lends interest and stimulation, but it is also the case that students need variety (even though pigeons may not). Consequently, instead of searching for a single best mode of presentation (scrambled book, linear, branched, etc.), we deliberately vary our presentations. A given programmed text may incorporate several formats. We are encouraged to use that format which appears to be most suited to the particular topic; if no standard format exists, we invent one. A cooperating teacher in one of our experimental centers reported, "The experimental class clapped when they saw the new books. I must have become very boring." Our mode of operation was to have live teaching for a few days alternate with programmed teaching. [It was a practical expedient since we did not have the time to convert our entire course into programmed materials.] This procedure had interesting results. It happened that by the time the students had finished working in a programmed text, they were eager to get back to live teaching, and after a period of live teaching, they started asking for more programmed texts. So, instead of programmed texts replacing teachers, it turns out that they make students more appreciative of their teachers! This experience suggests that programmed texts may be used to give teachers adequate time for preparing their own live presentations.

There are, of course, many other uses of programmed materials in addition to alternating live and programmed instruction. These include the use of programmed texts for homework, thereby reducing paper-grading; make-up work for students who have missed several days of classwork; instruction for homebound students; and remedial (or, perhaps, enrichment) work to help take care of individual differences. At the present time, we do not advocate using programmed materials, exclusively, for an entire course. Programmed materials should be regarded as a teaching aid similar to those now available in a well-equipped classroom or audio-visual aids department.

Another goal for our programmers is to encourage students to discover the desired concepts by providing them with sequences of developmental exercises. In this respect we capitalize on the vast experience gained by UICSM writers and teachers during the decade of UICSM's existence. Although many programmers subscribe verbally to the "Socratic" method of teaching, one would think from looking at many of the available programs that their authors believed that Socrates taught by telling his students small bits of information and that his famous questioning technique consisted merely of asking, "Now what did I just tell you?" Our programming technique is to start with performance-type questions which gradually reveal a pattern. Our test for discovery is to give more performance-type questions but on a much higher difficulty level. Verbalization of the discovery is withheld at least until students give evidence that the discovered principle is an old friend. We feel that a student who can *do* a problem doesn't need a rule whereas merely having a rule does not guarantee successful application of it.

Our study should lead to the development of several principles of programming. These should be useful to teachers who wish to create programs of their own or who wish just to improve their own lesson-planning.

INTRODUCTION

The material in this section of the report is excerpted from programed instructional texts used by 580 students in the fall of 1962. Each part (programed booklet) deals with information contained in Unit I of UICSM High School Mathematics. For the most part, the programed material follows the sequence found in Unit I. There are places, however, in which pedagogical and psychological considerations forced a change in order or content. Since there were cases in which the students used the texts almost completely without teacher assistance, it was essential that the text anticipate some student difficulties that experienced teachers have learned will arise at particular points in the learning process.

The matter of review at appropriate times is one which presents unique problems in writing a text in this manner. The programer has to decide whether a particular student needs a review before he should continue with new material. The programer thus must try to determine the quantity of review needed. In traditional text writing, the author is free to insert review questions at any time. A teacher, knowing the particular situation, will select apropos questions. The programer must have some criterion for inserting these questions. In our texts, we have used several approaches. In some, the review questions are scattered throughout the part, and are not pointed out specifically as review. There are parts in which the programers have provided a drill section. These sections review previously learned concepts which are needed as a foundation for what will immediately follow.

The UICSM High School Mathematics units contain many developmental exercises. These permit the student to explore a new topic before he receives any formal instruction on it. Intuition plays a large part in helping the student "see" how to work these problems. The programers have included many exercises of this type in the programed texts. Naturally, the students must be provided with answers immediately so that he will know whether he has chosen the right direction. This means that a two page spread in a typical text might be expanded to twenty pages in a programed text.

The notion of immediate reinforcement is vital to all the programed texts. Most questions are answered on the next page and the student benefits from the immediate knowledge of the correctness of his response. At times, the programer provides the student with some hints that are meant to help him determine the correct answer for himself.

When a student has arrived at the particular point in his learning at which he will accept any instance of a given principle, the text introduces appropriate vocabulary. Thus, the name of a basic principle is not used until some time after the student has learned to use that principle. The rules for manipulating real numbers are completely learned before the students learn to verbalize them.

There are some techniques developed for programed texts which are not used in most textbooks. One of these is the "Branching Technique". This form, in brief, involves a choice of pages for the student to read next, depending on which answer he gave to a particular question. When he answers a question, the student finds his answer among the three or four given in the text for that question. He is then directed to another page. If he has the correct answer he moved ahead or given an enrichment branch. An incorrect answer leads him to a page on which his misconception should be cleared up. In addition, a wrong answer leads to a remedial branch.

A second technique used in these parts is the "Zebra Technique". The format consists of a series of horizontal stripes, usually five, on each page. The stripes have two divisions, answer and question. The student reads a question, answers it, then turns the page. In the corresponding stripe on the page to which he turns he reads the correct answer in the first division. When he has checked his answer, he moves to the question beside this answer. This process continues until he is told to move to a different stripe. He proceeds through each stripe the same way.

A great amount of care has gone into the choosing of visual devices to aid the student. Eye-catching pictures and unusual print have been used throughout the texts. Certain items are used to prevent the waning of interest in tedious situations.

The volume of material contained in these texts may seem quite large, but, in practice, the eighteen parts take only eight to ten weeks to complete. The time span may be decreased by classroom discussion of the material. In conjuntion with this, the parts are used for homework assignments. So far, the work has been done mostly under experimental conditions, so its application to a normal situation has not been completely ascertained.

A work sheet has been prepared in conjunction with each part. Such a sheet permits the student to record his answers with a minimum of writing and to indicate which of his answers are correct.

The work sheet is intended to be more than just a place to record answers. The sample page from the work sheet for Part 114.5 illustrates three of its functions.

(1) The chart on page 53 is designed to be a discovery exercise. It is reproduced in total on the work sheet to allow the student to see the relationship between the last two columns in the chart.

(2) A "cue" is provided to aid the student in filling the blanks for page 54. "Cues" are useful devices on difficult exercises, exercises having several possible answers, and exercises in which we wish to develop a certain form for the answers.

(3) The Result-check column provides a record of the student's work. If he gets all of a given set of exercises correct, he circles the 'OK'. If he misses one or more of a set, he circles only the letters which correspond to his correct answers.

Check your answers.

	first number	second number	$\mathcal{O}$(first number)	$\mathcal{O}$(second number)	(first number) ÷ (second number)	$\mathcal{O}$(first number) ÷ $\mathcal{O}$(second number)
(a)	$^{+}5$	$^{-}3$	$^{-}5$	$^{+}3$	$^{+}5 \div {}^{-}3 = \frac{^{-}5}{3}$	$^{-}5 \div {}^{+}3 = \frac{^{-}5}{3}$
(1)	$^{-}6$	$^{-}2$	$^{+}6$	$^{+}2$	$^{-}6 \div {}^{-}2 = {}^{+}3$	$^{+}6 \div {}^{+}2 = {}^{+}3$
(2)	$^{+}7$	$^{+}5$	$^{-}7$	$^{-}5$	$^{+}7 \div {}^{+}5 = \frac{^{+}7}{5}$	$^{-}7 \div {}^{-}5 = \frac{^{+}7}{5}$
(3)	$^{-}8$	$^{+}9$	$^{+}8$	$^{-}9$	$^{-}8 \div {}^{+}9 = \frac{^{-}8}{9}$	$^{+}8 \div {}^{-}9 = \frac{^{-}8}{9}$
(4)	$\frac{^{+}1}{8}$	$^{-}2$	$\frac{^{-}1}{8}$	$^{+}2$	$\frac{^{+}1}{8} \div {}^{-}2$ $= \frac{^{+}1}{8} \times \frac{^{-}1}{2} = \frac{^{-}1}{16}$	$\frac{^{-}1}{8} \div {}^{+}2$ $= \frac{^{-}1}{8} \times \frac{^{+}1}{2} = \frac{^{-}1}{16}$

Record your results.

* * *

Fill the blanks.

	first number	second number	$\mathcal{O}$(first number)	$\mathcal{O}$(second number)	(first number) ÷ (second number)	$\mathcal{O}$(first number) ÷ $\mathcal{O}$(second number)
(1)	$^{-}4$	$^{+}2$	?	?	?	?
(2)	$^{+}4$	$^{-}3$	?	?	?	?
(3)	$^{-}16$	$^{-}5$	?	?	?	?
(4)	0	$^{+}182$	?	?	?	?

Check your answers.

	first number	second number	$\mathcal{O}$(first number)	$\mathcal{O}$(second number)	(first number) ÷ (second number)	$\mathcal{O}$(first number) ÷ $\mathcal{O}$(second number)
	$^-4$	$^+2$	$^+4$	$^-2$	$\frac{^-4}{^+2} = {}^-2$	$\frac{^+4}{^-2} = {}^-2$
	$^+4$	$^-3$	$^-4$	$^+3$	$\frac{^+4}{^-3} = -\frac{4}{3}$	$\frac{^-4}{^+3} = -\frac{4}{3}$
	$^-16$	$^-5$	$^+16$	$^+5$	$\frac{^-16}{^-5} = +\frac{16}{5}$	$\frac{^+16}{^+5} = +\frac{16}{5}$
	0	$^+182$	0	$^-182$	$\frac{0}{^+182} = 0$	$\frac{0}{^-182} = 0$

Record your results.

* * *

The exercises you have just worked suggest the following property of the real numbers:

If you pick a first real number and a ___?___ second real number,

the first number DIVIDED BY the ___?___ number

gives the same result as

the OPPOSITE of the ___?___ number DIVIDED BY the ___?___ of the second number.

Fill the blanks on your work sheet.

WORK SHEET

Part 114.5

	Page	Time
Start		
End		

ID No. _____

Answers | **Result-check**

Page 53

(1)	$^-4$	$^+2$	(a)	(b)	(c)	(d)
(2)	$^+4$	$^-3$	(a)	(b)	(c)	(d)
(3)	$^-16$	$^-5$	(a)	(b)	(c)	(d)
(4)	0	$^+182$	(a)	(b)	(c)	(d)

Page 53

1a b c d

2a b c d

3a b c d

4a b c d OK

Page 54

If you pick a first real number and a non (a) second real number,

the first number DIVIDED BY the _____ (b) number

gives the same result as

the OPPOSITE of the _____ (c) number

DIVIDED BY the op (d) of the second number.

Page 54

a

b

c

d OK

Page 55

(1)	$^-28$	$^-7$	(a)	(b)	(c)
(2)	$^+99$	$^+11$	(a)	(b)	(c)
(3)	$^-15$	$^+\frac{1}{3}$	(a)	(b)	(c)
(4)	$^+\frac{1}{2}$	$^-4$	(a)	(b)	(c)

Page 55

1a b c d

2a b c d

3a b c d

4a b c d OK

PART 101

Content

Part 101 deals with the topics which are found in the Introduction of Unit I, UICSM High School Mathematics. The Introduction is concerned with the notion that a symbol and its referent are not the same. In order to make this distinction clear to the reader, the text always uses single quotes around the symbol when referrin to the symbol. There is no provision for remedial work in this part. Remedial type questions are asked in later sections of the text.

Approach

Part 101 uses a sequence of frames containing brief questions which are designed to build on the students' past experience. Since this notion is a new mode of thought for the student, he needs much practice in applying it to various situations Finally, he is led to the point where he finds that a number and its numerals are different, and then he is given a sequence in which this notion is strengthened.

Self-Testing Instruments

This text contains twelve quizzes to be taken by the student. He takes the quiz anc checks his answers with the key on the next page. When he has finished his self-test, he is informed (by the text) of what frame to read next. A sample quiz is shown in this resumé as it would appear in the text.

Part 101 Sample Pages

On the next pages is a sample of the pages in Part 101. Since it is a sample of 14 frames chosen from 389 frames, the content is necessarily abbreviated. In this selection, it is doubtful that the many notions developed in Part 101 will be evident to a reader of this resumé, but we have tried to show how the vital notions of numeral vs. number are treated by this text. We suggest that the reader follow the frames as a student would and answer the questions.

The "Zebra" Technique

This form of programed material involves the use of a series of stripes across each page of the text. The stripes are alternately shaded and unshaded. The reader first reads the contents of the top stripe on succeeding pages. Then he proceeds to the second stripe through the book, then the third, and similarly through each level until he has read each frame. Each stripe on a page contains an answer and a question, the answer is for the previous page's question. There is a dividing line between this answer and the next question.

	Symbols like '1', '5', '10', '50', '100', '1000', and '1962' are Arabic numerals. Symbols like 'I', 'V', 'X', 'L', 'M' and 'MCMLXII' are Roman _____. [1]
[3] numerals	We can see all 13 stripes on a flag of the U.S.A. but we can't see the number 13. We can see a '13' but a '13' is merely a _____ f that names 13. [4]
[6] cannot	Whenever a person writes about numbers, he puts marks on paper. When he talks about numbers, he makes certain sounds. But the marks on paper and the sounds _____ (are, are not) the numbers themselves. [7]
[9] (a) equivalent (b) numerical (c) expressions	Punctuate this sentence using single quotes on your work sheet. 6 is a number and a digit 6 is a placeholder. Similarly, 0 is a number and a digit 0 is a placeholder. [10]
[12] The numeral '5 + 6' is not the numeral '6 + 5', but the number 5 + 6 is the number 6 + 5.	Punctuate this sentence on your work sheet. 33 is a 2-digit numeral which is equivalent to the numeral 29 + 4. [13]

[1] numerals	Numerals are not numbers. Numerals are names of _____. [2]
[4] numeral	We can see 2 shoes but we can't see the number 2. We can see a '2', but that is only a numeral for the number _____. [5]
[7] are not	Since '2 + 1' is equivalent to '7 - 4' we may say that '2 + 1' and '7 - 4' are equivalent numerical expressions. Since '3 × 2' is not equivalent to '36 ÷ 4', we may say that '3 × 2' and '36 ÷ 4' _____ (are, are not) equivalent numerical expressions. [8]
[10] 6 is a number and a digit '6' is a placeholder. Similarly, 0 is a number and a digit '0' is a placeholder.	Punctuate this sentence on your work sheet. 329 is a 3-digit numeral. [11]
[13] '33' is a 2-digit numeral which is equivalent to the numeral '29 + 4'.	Punctuate: I can't write 3 + 4, but I can write a 3 + 4 [14]

[2] numbers	Just as a thing and its names are different, a number and its n _____ s are different. [3]
[5] 2	You can point to each of seven objects, but you _____ (can, cannot) point to 7. [6]
[8] are not	(a) '1 + 1' and '2' are _____ numerical expressions. [9] (b) '5' and '2 + 3' are equivalent _____ expressions. (c) '8 × 2' and '19 - 3' are equivalent numerical _____.
[11] [The numeral] '329' is a 3-digit numeral	Punctuate this sentence on your work sheet. The numeral 5 + 6 is not the numeral 6 + 5, but the number 5 + 6 is the number 6 + 5. [12]
[14] I can't write 3 + 4 but I can write a '3 + 4'. Now turn to QUIZ 1 on the next page and take it.	

This quiz is taken as soon as the student has reached the frame that cues him to turn to this quiz. When he has completed the quiz, he turns to the next page on which the answers appear.

Quiz I. Punctuate each sentence with single quotes so that it becomes true.

(1) Sammy erased an 8 and wrote a 7 in its place.

(2) 33 is a 2-digit numeral.

(3) 7 + 8 and 30 ÷ 2 are equivalent numerical expressions.

(4) 3 + 2 is not a number but 9 - 4 is.

(5) 7 × 3 is not a numeral but 63 ÷ 3 is.

[Answers on next page]

Answers for QUIZ I

(1) '8'
'7'

(2) '33'

(3) '7 + 8'
'30 ÷ 2'

(4) '3 + 2'

(5) '63 ÷ 3'

When the student has read the answers and checked his responses against these, he proceeds to the frame indicated at the end of the quiz or returns his text to his teacher if it is the last quiz. There is no remedial work in this text. The text contains many frames asking review questions to strengthen ideas developed earlier.

In the next section of this report we shall present Parts 102, 103, and 104. The following are sample items from the tests of these parts. They will supply the reader with some of the objectives of these three parts. The main objective of the three parts is to give the student a reasonable degree of competency in performing the operations addition and multiplication on pairs of real numbers.

Cedric is running in a way that would carry him up a regular stairway at 6 steps per second. So is Alfred. Cedric is chasing Alfred, who is 6 steps ahead. They are running on a very long escalator, whose steps are moving down at a rate of 3 steps per second. How long will it take Cedric to catch Alfred?

A) Cedric will catch Alfred in 6 seconds.
B) Cedric will catch Alfred in 2 seconds.
C) Cedric will catch Alfred in 1 second.
D) Cedric will not catch Alfred.

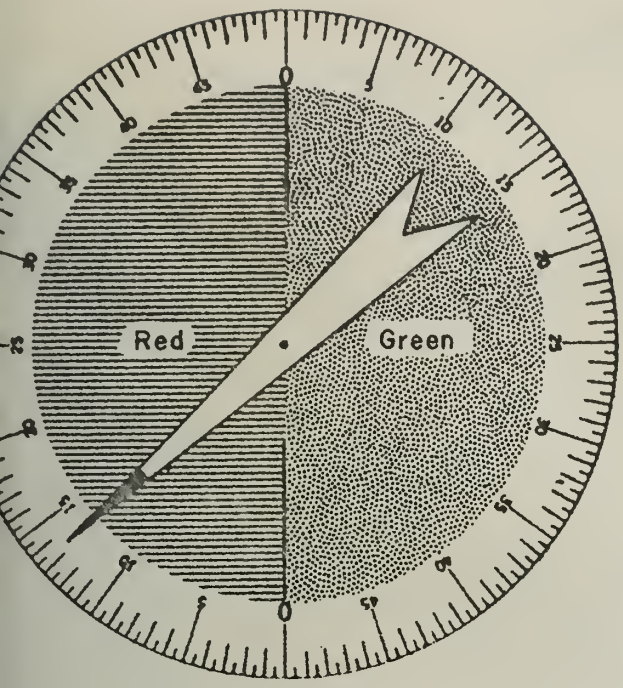
Penelope asked Arthur to meet her at eight p. m., on a certain night. He was to meet her on the street corner which is six blocks from the center of town along Main Street. He did not show up. The next day Penelope scolded him for not meeting her. Arthur replied that he had gone to the appointed place at eight p. m., and had waited for her for an hour. Could he have been telling the truth?

Bruce was playing a game with Eric. On the first play Bruce gained 2 points. On following plays he gained 5, lost 8, gained 13 lost 10, lost 5, gained 1. In order for Brue to just "break even", what must the outcome of the last play be? (Assume that gains are measured by positive real numbers.)

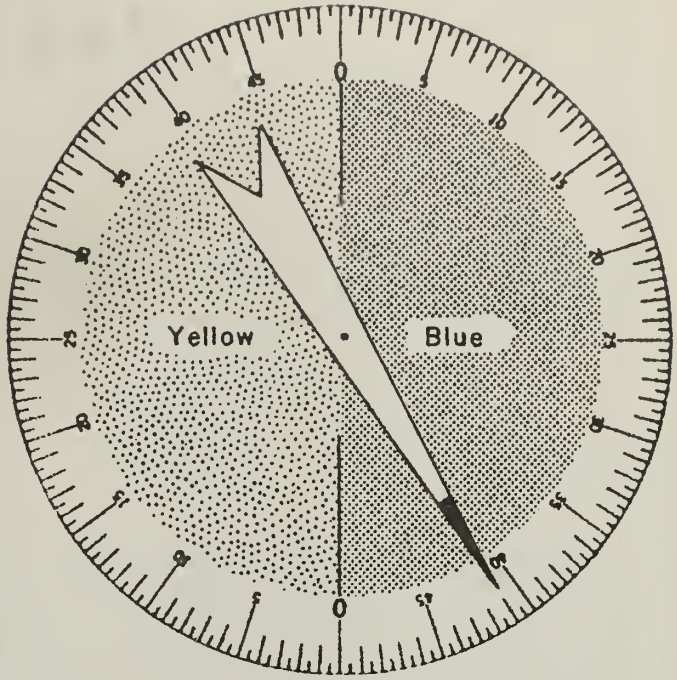
Each of the following problems contains a list of real numbers. Your task in each case is to find the sum of the numbers listed.

$$^{-}31,\ ^{+}20,\ ^{-}11,\ ^{-}8\tfrac{1}{3}$$

$$^{+}17,\ ^{+}21,\ ^{+}4$$



First Spinner [See Problem 23.]



Second Spinner

Rexo is a game played with a pair of spinners like those shown above. To take his turn a player first spins one spinner and then the other.

Christina spun the first spinner and the pointer came to rest on the '15' on the green side. [We say that she got a green-fifteen.] Then she spun the second spinner and got a blue-seven. The result of these spins was an increase of 105 points in her score.

When Jon spun, he got a red-30 on the first spinner and a blue-10 on the second. Jon lost 300 points on that turn.

The table below gives the essential information for these and several more turns. Study the information given and then fill in the blanks in the last four rows of the table. [According to the rules of the game, if the pointer comes to rest on a '0', it is considered as lying on a '0' rather than as lying on a '50'.]

green-15	blue-7	105-point increase
red-30	blue-10	300-point decrease
red-12	blue-12	144-point decrease
red-45	yellow-2	90-point increase
green-11	yellow-11	121-point decrease
red-9	yellow-20	180-point increase
green-40	blue-7	280-point increase
green-15	yellow-30	450-point decrease
red-13.5	blue-41	_____
red-17	yellow-6.5	_____
green-13	_____	169-point decrease
_____	blue-9	153-point increase

PART 102

Part 102 is concerned with the beginning treatment of real numbers as directed trips on a road. Pages 1-6 of Unit I, UICSM High School Mathematics, contain most of the material programed in this part.

Approach

Part 102 uses some pages for pure exposition. This is usually followed by several pages of questions on the material explained on that page. The questions are answered on the next page in most cases. In a few cases, the students are led to one of several different pages according to which answer they give to the question.

The first set of pages develops the idea of taking a single trip along a road. This idea will be useful later in naming real numbers. It gives the student a model for real numbers which will be readily exploited when needed.

[Part 102] [Page 14]

NUMERALS FOR REAL NUMBERS

In order to work with real numbers, we need names for them. We are going to describe a system for writing numerals for real numbers. These numerals are not the ones that most other people use, but they are helpful to start with. As soon as we learn a little bit more about real numbers, we'll switch to the numerals that other people use. Naturally, the real numbers themselves will not change when we make the switch---only their names will.

We said that real numbers are numbers which can be used to measure trips. They can be used for both the distance between the beginning and end points of a trip and the direction from the beginning point to the end point. So, a numeral for a real number should have a part which gives distance, and a part to tell direction.

Since our "old" numbers of arithmetic are perfectly suited for distance, we shall use their NAMES as part of the numerals for real numbers. The rest of the numeral, the part used to indicate direction, will be an arrow.

Here are some examples of numerals for real numbers:

$\vec{5}$ $\overleftarrow{8}$ $\overleftarrow{4}$ $\vec{7}$

The real numbers

$\vec{2}$ and $\vec{3}$

are measures of trips made in the same direction. Which of these two trips has the greater distance?

Circle the correct answer on your work sheet.

Check your answer.

You were right if you thought that a trip measured by $\vec{3}$ has greater distance than one measured by $\vec{2}$. As you know, the '3' of '$\vec{3}$' and the '2' of '$\vec{2}$' tells us the distances for these two trips, and 3 is greater than 2.

Record your result.

* * *

The real numbers $\overleftarrow{5}$ and $\overleftarrow{7}$ are also measures of trips made in the same direction, but this direction is the opposite of the direction for the trips measured by $\vec{2}$ and $\vec{3}$. Do you see that $\overleftarrow{7}$ measures a trip of greater distance than does $\overleftarrow{5}$?

Consider trips measured by the real numbers $\overleftarrow{5}$ and $\overleftarrow{3}$, respectively. Which of these trips has the greater distance? Since the distance traveled in a $\overleftarrow{5}$-mile trip is 5 miles, while the distance traveled in a $\overleftarrow{3}$-mile trip is 3 miles, it follows that a trip of $\overleftarrow{5}$ miles is __?__ than one of $\overleftarrow{3}$ miles.

Answer this question on your work sheet.

Check your answer.

The distance traveled in a $\overleftarrow{5}$-mile trip is 5 miles.

The distance traveled in a $\overleftarrow{3}$-mile trip is 3 miles.

So, a trip of $\overleftarrow{5}$ miles is *longer* than one of $\overleftarrow{3}$ miles.
(or: *of greater distance*)

Record your result.

* * *

Answer the following questions on your work sheet.

(1) If Bill and Edgar start at the same point and Bill takes a trip of $\vec{2}$ miles and Edgar a trip of $\overleftarrow{2}$ miles, they will be __?__ miles apart at the end of their trips.

(2) If Bill and Edgar start at the same point and both take trips of $\vec{2}$ miles, they will be __?__ miles apart at the end of their trips.

Check your answers.

(1) Bill and Edgar will be *4* miles apart at the end of their trips, when they go in opposite directions.

(2) Bill and Edgar will be *0* miles apart at the end of their trips, when they go in the same direction.

Record your results.

* * *

Whenever you learn a new name for something, it is natural to ask how to pronounce the name, in case you wish to use it in conversation. Let's agree to pronounce names like:

$\vec{2}$ and: $\vec{4}$

as you would pronounce 'right two' and 'right four'. As you might guess, we pronounce:

$\overleftarrow{5}$ and: $\overleftarrow{9}$

as you would pronounce 'left five' and 'left nine'.

On your work sheet, write word phrases which show how to pronounce the following numerals.

(1) $\vec{7}$ (2) $\overleftarrow{6}$

(3) $\overleftarrow{4}$ (4) $\vec{1}$

(5) $\overleftarrow{97}$ (6) $\overleftarrow{19}$

Check your answers.

(1) *right seven* (2) *right six*

(3) *left four* (4) *right one*

(5) *right ninety-seven* (6) *left nineteen*

Record your results.

* * *

Fill the blanks on your work sheet.

(1) '$\vec{2}$' is pronounced as '__?__ two'.

(2) '__?__' is pronounced as 'left seven'.

(3) '__?__' is pronounced as 'left one half'.

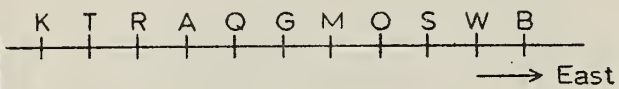
(4) '$\vec{4\frac{1}{2}}$' is pronounced as 'right __?__ and one half'.

(5) '$\vec{3\frac{1}{2}}$' is pronounced as 'right __?__ and __?__'.

(6) '$\overleftarrow{2\frac{1}{4}}$' is pronounced as 'left __?__ and __?__ fourths'.

Be sure that you have worked all of the exercises you can. Then,

[Part 102] [Page 20]



Now, we are ready to do some problems in which we use real numbers to measure trips made in one of two opposite directions. Suppose that the trips are made along the road pictured above. The two opposite directions are east and west. Let us AGREE to use real numbers like $\overrightarrow{6}$ and $\overrightarrow{3}$ to measure trips made in the easterly direction. So, for example, $\overrightarrow{2}$ measures a 2-miles-to-the-east trip, say, a trip from T to A. Then, $\overleftarrow{2}$ measures a 2-miles-to-the-west trip, say, a trip from B to S.

What real number measures the trip from A to M?

Circle the correct answer on your work sheet.

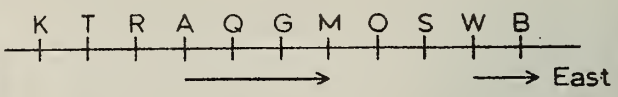
$\overrightarrow{4}$ $\overrightarrow{3}$ $\overleftarrow{3}$

If you think the answer is:

- $\overrightarrow{4}$ . . . . . .Turn to PAGE 23.
- $\overrightarrow{3}$ . . . . . .Turn to PAGE 22.
- $\overleftarrow{3}$ . . . . . .Turn to PAGE 21.

[Part 102] [Page 21]

Check your answer.

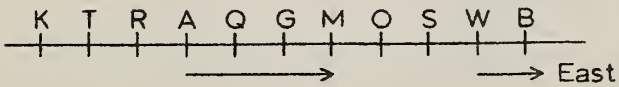


You said that the trip from A to M is measured by $\overleftarrow{3}$. Did you forget that on Page 20 we agreed that we would use real numbers like $\overrightarrow{3}$ to measure trips made in the easterly direction? The road map shows us that M is east of A. So, to get from A to M, we make a trip to the east. [You might argue that one could get from A to M by traveling west and going around the world. That's a good point, but let's agree that by a 'trip' we shall mean the shortest and most direct one.]

Turn back to PAGE 20 and try the question again.

[Part 102] [Page 22]

Check your answer.



We agreed to use real numbers like $\overrightarrow{6}$ to measure trips in the easterly direction.

You said that the trip from A to M is measured by $\overrightarrow{3}$. You are correct.

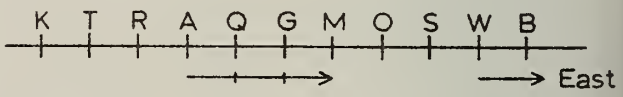
Now, give the real numbers which measure the trips listed below. Write the answers on your work sheet. [Notice that the problems go across the page.]

(1) R to Q	(2) T to G	(3) O to G
(4) B to A	(5) A to B	(6) G to W

When you have answered all the questions, turn to PAGE 24.

[Part 102] [Page 23]

Check your answer.



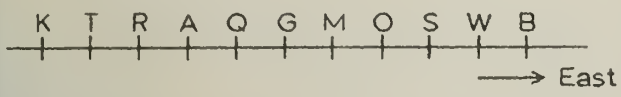
We agreed to use real numbers like $\overrightarrow{3}$ to measure trips in the easterly direction.

You said that the trip from A to M is measured by $\overrightarrow{4}$. Put your finger on the map at marker A and move your finger slowly in the easterly direction. When you reach marker Q, you've traveled 1 mile. Keep going, and when you reach marker G, you've traveled a a total of 2 miles. You have only one mile to go before you reach the end point of the trip from A to M. The trip from A to M is not measured by $\overrightarrow{4}$.

Turn back to PAGE 20 and try the question again.

[Part 102] [Page 24]

Check your answers.



(1) R to Q $\vec{2}$ (2) T to G $\vec{4}$ (3) O to G $\overleftarrow{2}$

(4) B to A $\overleftarrow{7}$ (5) A to B $\vec{7}$ (6) G to W $\vec{4}$

Record your results.

Did you notice that two of the trips described above are measured by the real number $\vec{4}$? There are other trips for this road which are measured by $\vec{4}$. Here are some others besides T to G and G to W.

K to Q R to M A to O

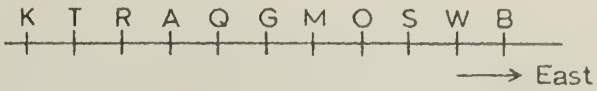
Q to S M to B

Are there still more? Circle the 'Yes' or the 'No' on your work sheet.

If you said that the answer is: YES, turn to PAGE 26. / NO, turn to PAGE 25.

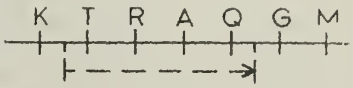
[Part 102] [Page 25]

Check your answer.

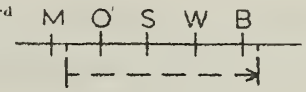


There are many more trips on this road which can be measured by the real number $\vec{4}$. In answering 'No' you may have been assuming that a trip had to start and end at the mile markers. The only difficulty with trips which do not start and end at a mile marker is that they are not as easy to describe. But, it can be done. For example, here are some complicated descriptions of other trips measured by $\vec{4}$:

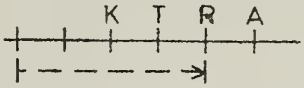
The trip from the point halfway between K and T to the point halfway between Q and G



The trip from the point one third of a mile east of M to the point one third of a mile east of B



The trip from the point two miles west of K to the point R



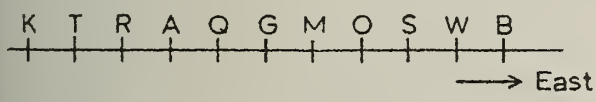
Of course, for the last two trips descriptions to make sense, we would have to assume that the road continues east of B and west of K.

Record your result.

Turn to PAGE 27.

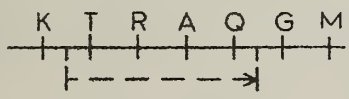
[Part 102] [Page 26]

Check your answer.

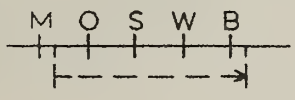


Yes, there are more trips on this road measured by the real number $\vec{4}$. In fact, there are infinitely many. You probably realized that trips on this road do not have to begin and end at the mile markers. Here are some descriptions of other trips measured by $\vec{4}$:

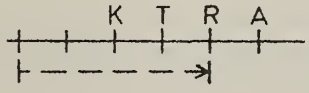
The trip from the point halfway between K and T to the point halfway between Q and G



The trip from the point one third of a mile east of M to the point one third of a mile east of B



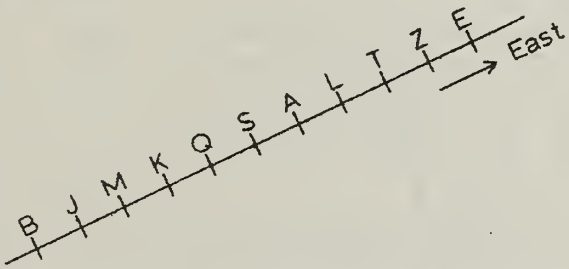
The trip from the point two miles west of K to the point R



Of course, for the last two trip descriptions to make sense, we would have to assume that the road kept going east of B and west of K.

Record your result.

[Part 102] [Page 27]



In giving the problems on this page, we shall abbreviate a trip description such as:

the trip from S to M

by just writing:

S, M

Now, for trips along the road pictured above, let's agree that real numbers such as $\vec{2}$ and $\overleftarrow{5}$ measure trips in the direction from say, B to E. Then, for example,

$\vec{3}$ measures S, M and also T, S

while

$\overleftarrow{4}$ measures Q, T and also M, A.

Write answers to the following questions on your work sheet.

Circle the descriptions of trips which are measured by $\vec{5}$

1. B, S 2. K, T 3. Z, Q 4. S, E 5. M, T 6. J, A

Circle the descriptions of trips which are measured by $\overleftarrow{5}$.

1. S, B 2. T, K 3. Q, Z 4. E, S 5. T, M 6. A, J

The second set of pages shows the way we lead into the notion of adding real numbers. Here is the major purpose of this model [trips along a raod] for real numbers. It will also be available to the student when he is working with the abstract notions presented later.

[Part 102] [Page 57]

Consider the pair of real numbers, $\overrightarrow{3}$ and $\overrightarrow{5}$. Suppose that the first of these measures a trip starting at a point A, going in the easterly direction, and ending at point B. And, suppose that $\overrightarrow{5}$ measures a second trip starting at point B [that is, the end point of the first trip] and ending at point C.

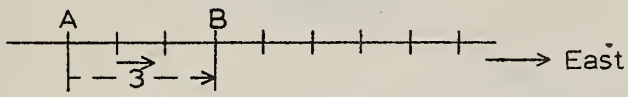
Now, what real number measures the single trip which would take you from A, the starting point of the first trip, to C, the end point of the second trip?

Write the answer in your work sheet.

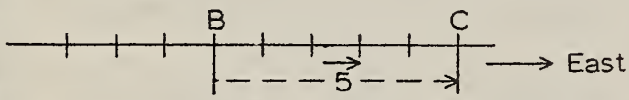
[Part 102] [Page 58]

Check your answer.

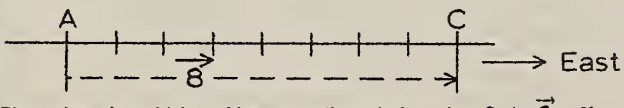
We are asked to consider the pair of real numbers, $\overrightarrow{3}$ and $\overrightarrow{5}$. Suppose that the first of these measures a trip starting at a point A and ending at point B, which is 3 units to the east.



And, suppose that $\overrightarrow{5}$ measures a trip starting at point B [that is, the end point of the first trip] and ending at point C, five units to the east of B.



Now, what real number measure the SINGLE trip from A, the starting point of the first trip, to C, the end point of the second trip?



The real number which would measure that trip from A to C, is $\overrightarrow{8}$. If you missed this last exercise, turn to PAGE 59. If you answered it correctly, read the paragraph below.

For convenience in writing about a pair of successive trips like these, we shall use the notation '$(\overrightarrow{3}, \overrightarrow{5})$'. To show that such a pair of trips takes you to the same place as a single trip measured by $\overrightarrow{8}$, we shall write:.

$$(\overrightarrow{3}, \overrightarrow{5}) \ldots\ldots \overrightarrow{8}$$

The pair of successive trips $(\overrightarrow{2}, \overrightarrow{6})$ could be replaced by a single trip measured by _?_.

Record the answer on your work sheet.

Turn to PAGE 63.

[Part 102] [Page 63]

Check your answer.

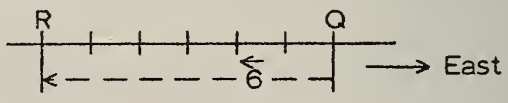
$$(\overrightarrow{2}, \overrightarrow{6}) \ldots\ldots \overrightarrow{8}$$

Record your result.

* * *

So far, we have discussed successive trips which are made in the same direction. It is also possible to consider successive trips made in opposite directions.

For example, consider the pair of real numbers $(\overleftarrow{6}, \overrightarrow{2})$ and the pair of successive trips which they measure. Suppose that the first number measures



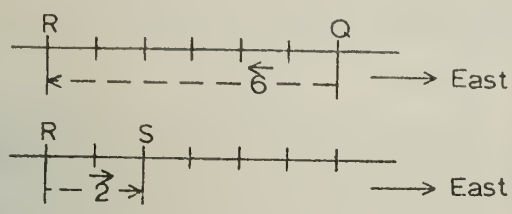
the trip from, say, Q to R. Now, think of the second number, $\overrightarrow{2}$, as measuring a trip which starts at R and ends at, say, S.

Mark the point S on your work sheet.

[Part 102] [Page 64]

Check your answer.

$(\overleftarrow{6}, \overrightarrow{2})$



Record your result.

* * *

Now, let's continue with the problem above.

Answer the following questions on your work sheet.

(1) What real number measures the trip from Q to R?

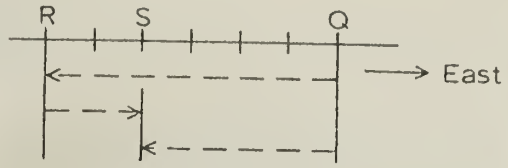
(2) What real number measures the trip from R to S?

(3) What real number measures the trip from Q to S?

(4) $(\overleftarrow{6}, \overrightarrow{2})$ ..... ? .

[Part 102] [Page 65]

Check your answers.



(1) What real number measures the trip from Q to R? $\overleftarrow{6}$

(2) What real number measures the trip from R to S? $\overrightarrow{2}$

(3) What real number measures the trip from Q to S? $\overleftarrow{4}$

(4) $(\overleftarrow{6}, \overrightarrow{2})$ ..... $\overleftarrow{4}$.

Record your results.

* * *

Now, consider the pair of real numbers $(\overleftarrow{3}, \overrightarrow{7})$. Suppose that the first, $\overleftarrow{3}$, measures a trip to the west, and the second, $\overrightarrow{7}$, measures a trip starting at the end point of the first and traveling east. What single real number could replace this pair, and measure the direct trip to the same place. [Make a diagram if it helps you to solve the problem.]

Circle the correct answer on your work sheet.

$\overrightarrow{4}$ $\overleftarrow{10}$ $\overleftarrow{4}$

If your answer is:
- $\overrightarrow{4}$ ..... turn to PAGE 66.
- $\overleftarrow{10}$ ..... turn to PAGE 68.
- $\overleftarrow{4}$ ..... turn to PAGE 67.

The third set of pages shows an "applied" problem using real numbers in a profit and loss situation. Changing the context helps develop the abstractions of real numbers.

[Part 102] [Page 75]

Check your answer.

We agreed that a real number like $\overrightarrow{3}$ would be used to indicate the receiving of money. The school supply store received $27.00 for the items it sold, so we would record this as $\overrightarrow{27}$.

* * *

The Student Council of Zabranchburg High School held its annual Fall Fair in order to raise money for the school activity fund. Eileen decided to sell candied apples. The council gave her $5.00 to buy supplies. She spent $2.00 to decorate her booth and $3.00 to buy apples and the ingredients to make the candy. So, when the fair opened she had spent $5.00. The real number which measures the money she spent is $\overleftarrow{5}$. On the first day Eileen took in $9.00 which we would measure as $\overrightarrow{9}$.

To find the outcome of the first day's business, we treat Eileen's purchases and sales just as we treated successive trips. That is, we write:

$$(\overleftarrow{5}, \overrightarrow{9}) \text{ ..... } \overrightarrow{4}$$

The outcome of the first day's business is measured by $\overrightarrow{4}$. This means that Eileen had a $4.00 profit.

To prepare for the second day of the fair, Eileen spent $6.00 on supplies. What real number represents Eileen's $6.00 purchase?

Check your answer.

Eileen spent $6.00. We measure this by $\overleftarrow{6}$.

Record your result.

* * *

Unfortunately, it rained on the second day of the fair. Very few visitors came and Eileen took in only $3.00. The record for the day looked like this:

Expenses ..... $\overleftarrow{6}$

Sales ..... $\overrightarrow{3}$

Outcome ..... $(\overleftarrow{6}, \overrightarrow{3})$ ..... $\overleftarrow{3}$.

The outcome was measured by ___?___. This means that the candied apple booth had a ___?___ of ___?___.

Fill in the blanks on your work sheet.

Check your answers.

The outcome of a day's business was measured by $\overleftarrow{3}$. This means that the candied-apple booth had a loss of $3.00.

Record your results.

* * *

The table below is meant to show all of the transactions involving the candied-apple booth during the week of the fair. Complete this table on your work sheet.

DAY	EXPENSES	SALES	CALCULATIONS OF OUTCOME
MONDAY	$\overleftarrow{5}$	$\overrightarrow{9}$	$(\overleftarrow{5}, \overrightarrow{9})$ ..... $\overrightarrow{4}$
TUESDAY	$\overleftarrow{6}$	$\overrightarrow{3}$	$(\overleftarrow{6}, \overrightarrow{3})$ ..... $\overleftarrow{3}$
WEDNESDAY	$\overleftarrow{6}$	$\overrightarrow{11}$	$(\overleftarrow{6}, \underline{?})$ ..... $\underline{?}$
THURSDAY	$\underline{?}$	$\overrightarrow{7}$	$(\overleftarrow{8}, \overrightarrow{7})$ ..... $\underline{?}$
FRIDAY	$\overleftarrow{7}$	$\underline{?}$	$(\underline{?}, \overrightarrow{12})$ .... $\underline{?}$
SATURDAY	$\overleftarrow{9}$	$\underline{?}$	$(\underline{?}, \underline{?})$ ..... $\overrightarrow{5}$

Here is the completed table. Check your answers.

DAY	EXPENSES	SALES	CALCULATIONS OF OUTCOME
MONDAY	$\overleftarrow{5}$	$\overrightarrow{9}$	$(\overleftarrow{5}, \overrightarrow{9})$ ..... $\overrightarrow{4}$
TUESDAY	$\overleftarrow{6}$	$\overrightarrow{3}$	$(\overleftarrow{6}, \overrightarrow{3})$ ..... $\overleftarrow{3}$
WEDNESDAY	$\overleftarrow{6}$	$\overrightarrow{11}$	$(\overleftarrow{6}, \overrightarrow{11})$ .... $\overrightarrow{5}$
THURSDAY	$\overleftarrow{8}$	$\overrightarrow{7}$	$(\overleftarrow{8}, \overrightarrow{7})$ ..... $\overleftarrow{1}$
FRIDAY	$\overleftarrow{7}$	$\overrightarrow{12}$	$(\overleftarrow{7}, \overrightarrow{12})$ .... $\overrightarrow{5}$
SATURDAY	$\overleftarrow{9}$	$\overrightarrow{14}$	$(\overleftarrow{9}, \overrightarrow{14})$ .... $\overrightarrow{5}$

Record your results.

* * *

Answer these questions on your work sheet.

1. Did Eileen have a profit or a loss on Wednesday?
2. How much money did she spend on Friday?
3. How much money did she receive on Saturday?
4. The outcome of business on Monday was $\overleftarrow{4}$. The outcome on Tuesday was $\overleftarrow{3}$. So, for the two days taken together, the outcome was ___?___.
5. On which days did Eileen take a loss?
6. What was the total outcome after the 6 days of the fair?

Check your answers.

1. Did Eileen have a profit or a loss on Wednesday?
   $\overrightarrow{5}$ indicates a profit.
2. How much money did she spend on Friday?
   $\overleftarrow{7}$ indicates an expense of $7.00.
3. How much money did she receive on Saturday?
   $\overrightarrow{14}$ indicates the receipt of $14.00.
4. The outcome of business on Monday was $\overrightarrow{4}$. The outcome on Tuesday was $\overleftarrow{3}$. So, for the two days taken together, the outcome was $\overrightarrow{1}$.
5. On which days did Eileen take a loss? Tuesday and Thursday
6. What was the total outcome after the 6 days of the fair?
   By considering the outcome of each day's business as one of several successive trips we can use real numbers when figuring total assets for consecutive business days. The daily outcomes were as follows: Monday, $\overrightarrow{4}$; Tuesday, $\overleftarrow{3}$; Wednesday, $\overrightarrow{5}$; Thursday, $\overleftarrow{1}$; Friday, $\overrightarrow{5}$; Saturday, $\overleftarrow{5}$.

TOTAL OUTCOME:

after Monday $\overrightarrow{4}$ ..... $\overrightarrow{4}$

after Tuesday $(\overrightarrow{4}, \overleftarrow{3})$ ..... $\overrightarrow{1}$

after Wednesday $(\overrightarrow{1}, \overrightarrow{5})$ ..... $\overrightarrow{6}$

after Thursday $(\overrightarrow{6}, \overleftarrow{1})$ ..... $\overrightarrow{5}$

after Friday $(\overrightarrow{5}, \overrightarrow{5})$ ..... $\overrightarrow{10}$

after Saturday $(\overrightarrow{10}, \overleftarrow{5})$ ..... $\overrightarrow{15}$

So we see that after the 6 days of the fair the total outcome was $\overrightarrow{15}$. The booth made a profit of $15.00.

Record your results.

This is the end of Part 102.

PART 103

Content

1. Addition of real numbers
2. Subsets of real number system
   (a) Negative numbers, zero, positive numbers
   (b) Nonpositive reals and positive reals
3. Function machines

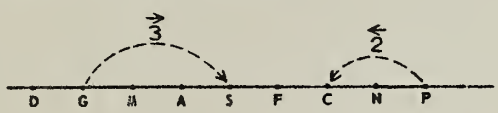
Approach

Aside from a "Zebra" section of 59 frames of practice on the arithmetic of fractions, the text is similar in orgainization to Part 102. There is an enrichment branch for the better student; for his slower counterpart there is additional help on the basic material.

The first set of pages establishes the raised plus sign for naming positive numbers, and the raised minus sign for naming negative numbers.

[Part 103] [Page 1]

POSITIVE AND NEGATIVE NUMBERS



You have been using "arrow" numerals to measure trips like these shown on the road map above:

The trip from G to S has measure $\overrightarrow{3}$.

The trip from P to C has measure $\overleftarrow{2}$.

The numerals:

$$\overrightarrow{3}\ ,\ \overleftarrow{2}\ ,\ \overrightarrow{3.1}\ ,\ \overleftarrow{2.7}\ ,\ \overrightarrow{107}\ ,\ \overleftarrow{\tfrac{3}{4}}$$

stand for real numbers, but there are other ways to write them. Here are some numerals that many people use. We shall use them in this course.

$$^{+}3\ ,\ ^{-}2\ ,\ ^{+}3.1\ ,\ ^{-}2.7\ ,\ ^{+}107\ ,\ ^{-}\tfrac{3}{4}$$

Pronounce '$^{+}3$' as 'POSITIVE three' and '$^{-}2$' as 'NEGATIVE two'.

Complete the following sentences on your work sheet. [Watch your spelling!]

(1) '$^{-}1$' is a numeral for the real number __?__ __?__.

(2) '$^{+}5$' is a numeral for the real number __?__ __?__.

(3) '$^{-}\frac{1}{2}$' is a numeral for the real number __?__ __?__.

[Part 103] [Page 2]

Check your answers.

(1) '$^{-}1$' stands for negative one.
(2) '$^{+}5$' stands for positive five.
(3) '$^{-}\frac{1}{2}$' stands for negative one half.

Record your results.

* * *

Here is how we write our "new" numerals for positive and negative real real numers:

POSITIVE REAL NUMBERS

A raised sign, '$^{+}$', followed by a numeral for a number of arithmetic.

Examples: $^{+}8$, $^{+}11$, $^{+}\frac{1}{5}$, $^{+}6\frac{1}{4}$, $^{+}9.03$

NEGATIVE REAL NUMBERS

A raised sign, '$^{-}$', followed by a numeral for a number of arithmetic.

Examples: $^{-}8$, $^{-}11$, $^{-}\frac{1}{5}$, $^{-}6\frac{1}{4}$, $^{-}9.03$

On your worksheet, use the raised-sign notation to write numerals for the given real numbers:

(1) negative five (2) positive seven (3) negative two thirds

[Part 103] [Page 3]

Check your answers.

(1) negative five: $^{-}5$
(2) positive seven: $^{+}7$
(3) negative two thirds: $^{-}\frac{2}{3}$

[Note: Do you have the '$^{+}$' and the '$^{-}$' in the correct position? If not, rewrite the answers correctly beside your wrong ones.]

Record your results on your work sheet

* * *

Here are some marks that some students make when they wish to refer to real numbers.

$+8$, $^{+}7$, 3 , 2^{+} , $^{-}4$, -5

As you can see, some of these are wrong because the sign is not in the correct position. These same marks are shown on your work sheet. Circle ONLY the marks there which are written CORRECTLY and use English words to tell how you would pronounce the ones written correctly.

The second set of pages gives a "practical" problem using the notions developed in Part 102, with the added feature of new notation from the first section of Part 103.

[Part 103] [Page 32]

Check your answers.

(1) $^{+}130$, $^{+}57$ . . . . $^{+}187$
(2) $^{-}118$, $^{+}142$ . . . . $^{+}24$
(3) $^{-}3527$, $^{+}199$ . . . . $^{-}3328$
(4) $^{-}0.025$, $^{-}0.775$ . . . . $^{-}0.8$

Record your results.

* * *

A FLOWER BUSINESS

Bill's father loaned him $10 to start a business during his summer vacation. His father asked him to repay the loan at the end of the summer. Bill decided to sell flowers at a stand on a busy intersection near his home.

Saturday morning he spent the entire $10 to buy flowers from a nurseryman. By noon he had sold them all for a total of $15. He decided to save the $15 and use it all for buying flowers on Monday morning. But the flower business was not so good on Monday, for by the end of the day his sales were only $11. The rest of the flowers had to be thrown away because they were wilted. So was Bill. Bill's story will continue after this question.

What is the DIFFERENCE between Bill's expenses and his sales (1) on Saturday? (2) On Monday? Fill the blanks on your work sheet using numbers of arithmetic.

[Part 103] [Page 33]

Check your answers.

		Expenses	Sales	Difference
(1)	Sat.	10	15	5
(2)	Mon.	15	11	4

Record your results.

* * *

Just telling the difference between Bill's expenses and his sales for each day does not give the whole story. We cannot learn just from the difference on Monday, for example, why Bill would be disappointed with the outcome of his business on that day. We need to know the direction as well as the difference of the outcome. This gives me the idea of using real numbers; how about you? Most people use positive numbers to measure profitable outcomes and negative numbers to measure losses.

Complete the table below on your work sheet, and this time use real numbers:

	Expenses	Sales	Outcome
Sat.	10	15	?
Mon.	15	11	?

[Part 103] [Page 34]

Check your answers.

	Exp.	Sales	Out-come
Sat.	10	15	+5
Mon.	15	11	-4

Of course we could have used "-5" and "+4", but we prefer to measure profits and losses with real numbers in the way most people do.

Record your results.

* * *

If Bill had quit the flower business on Saturday, he would have been able to repay his father and have a profit of $5 for himself.

If Bill had quit at the close of business on Monday, he still would have had enough money to repay his father. In fact, he would have earned a profit of $? for himself.

Refer to the table at the top of this page and write the answer on your work sheet.

[Part 103] [Page 35]

Check your answer.

If Bill had quit at the close of business on Monday, he still would have had enough money to repay his father and earn a profit of $ 1 for himself.

Record your result.

* * *

But Bill did not quit! He continued each day to spend the entire day's sales to purchase flowers from the nurseryman the next morning. The table below is a record of 7 days' business. Complete the "outcome" column:

	Exp.	Sales	Out-come
Sat.	10	15	+5
Mon.	15	11	-4
Tue.	11	12	?
Wed.	12	9	?
Thur.	9	13	?
Fri.	13	19	?
Sat.	19	25	?

Note that on pages 33 and 34 the text displays just two lines of the chart. This makes it possible to "pinpoint" the information the text is trying to get the student to understand.

At this point, there is some discussion of using the "road" technique for a profit and loss problem. Such a method of solution for this type of problem is gradually developed. The next pages complete this discussion.

[Part 103] [Page 36]

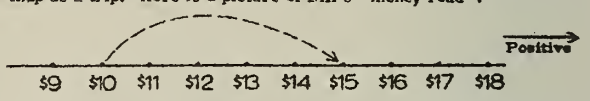
Check your answers.

	Exp.	Sales	Outcome
Sat.	10	15	+5
Mon.	15	11	-4
Tus.	11	12	+1
Wed.	12	9	-3
Thur.	9	13	+4
Fri.	13	19	+6
Sat.	19	25	+6

Record your results.

* * *

The amount of Bill's money changed each day. He started each day with a certain amount and ended that day with a certain amount. The amount at which the next day began was the same amount at which the previous day ended. The change in the amount may be shown on a road map as a trip. Here is a picture of Bill's "money road":

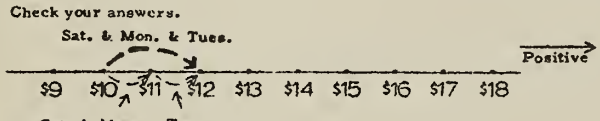


Let's say that on Saturday Bill took a trip on this money road from $10 to $15. The measure of the trip from $10 to $15 is the measure of the outcome of the business on Saturday. Fill this blank:

m($10, $15) = ?

[Part 103] [Page 39]

Check your answers.



(1) The direct trip for these two trips is from $10 to $12.

(2) The measure of this direct trip is +2.

(3) The sum of { the sum of the outcome-measures for Saturday and Monday (+1) and the outcome-measure for Tuesday (+1) } is +2.

(4) If Bill had quit the flower business at the close of Tuesday, he could have repaid his father and still had a profit of $2.

Record your results.

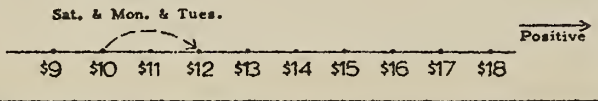
* * *

	Exp.	Sales	Outcome
Sat.	10	15	+5
Mon.	15	11	-4
Tue.	11	12	+1
Wed.	12	9	-3
Thur.	9	13	+4
Fri.	13	19	+6
Sat.	19	25	+6

As you can see from the table, Wednesday was nearly disastrous for Bill's business.

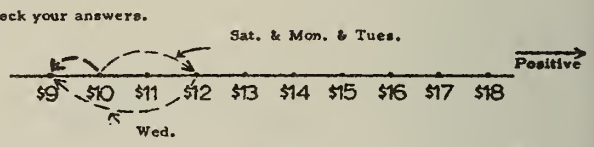
(1) On Bill's money road, shown below, the measure of the direct trip for Saturday & Monday & Tuesday and Wednesday is ?.

(2) If Bill had quit at the close of Wednesday, he would have been $? short of repaying his father.



[Part 103] [Page 40]

Check your answers.



(1) The measure of the direct trip for Sat. & Mon. & Tues. and Wed. is -1.

(2) If Bill had quit at the close of Wednesday, he would have been $1 short of repaying his father.

Record your results.

* * *

	Exp.	Sales	Outcome
Sat.	10	15	+5
Mon.	15	11	-4
Tue.	11	12	+1
Wed.	12	9	-3
Thur.	9	13	+4
Fri.	13	19	+6
Sat.	19	25	+6

To keep this story short, let's skip a few steps and get right to the end. The questions are the same as before, so try to answer them with as little work as possible.

(1) On Bill's money road, the measure of the direct trip for Sat. & Mon. & Tues. & Wed. & Thur. & Fri. and Saturday is ?.

(2) If Bill had quit at the close of the second Saturday of business, he could have repaid his father and still had a profit of $?.

[Part 103] [Page 41]

Check your answers.

(1) The measure of the direct trip for seven days is $^{+}15$.

(2) If Bill had quit at the close of the second Saturday, he could have repaid his father and still had a profit of $15. Let's say that he did just that, shall we?

Record your results.

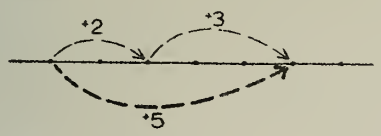
* * *

Do you remember . . .

(1) What the measures of two trips are called when the second trip begins where the first one ends?

(2) What the measure of the direct trip is called that begins where the first trip begins and ends where the second trip ends?

Here is an example of two trips with a direct trip.



On your work sheet, write what we call

(1) the measures $^{+}2$ and $^{+}3$.

(2) the measure $^{+}5$.

The third set of pages shows our use of the "function machine". Such a device gives the student a visual cue to enable him to do problems which, in standard form, would be fairly difficult.

[Part 103] [Page 69]

Check your answers.

(1)
$(^{+}7 + {}^{-}8) + {}^{-}3$
$= {}^{-}1 + {}^{-}3$
$= {}^{-}4$

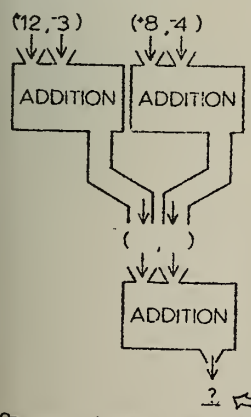
(2)
$(^{+}8 + {}^{-}3) + {}^{-}7$
$= {}^{+}5 + {}^{-}7$
$= {}^{-}2$

(3)
$^{+}4 + (^{-}7 + {}^{-}3)$
$= {}^{+}4 + {}^{-}10$
$= {}^{-}6$

Record your results.

* * *

Here is one way to show a certain addition problem:



And, here is another way of showing the same problem:

$(^{+}12 + {}^{-}3) + (^{+}8 + {}^{-}4) = \underline{?}$

On your work sheet write the answer to this problem.

[Part 103] [Page 70]

Check your answer.

$(^{+}12 + {}^{-}3) + (^{+}8 + {}^{-}4) = {}^{+}13$
$= {}^{+}9 + {}^{+}4$
$= {}^{+}13$

Record your result.

* * *

Write the answers for these problems on your work sheet.

(1) $(^{+}8 + {}^{-}2) + {}^{-}7 = \underline{?}$

(2) $(^{+}8 + {}^{-}2) + (^{-}3 + {}^{-}4) = \underline{?}$

(3) $^{+}2 + (^{+}6 + {}^{-}10) = \underline{?}$

(4) $(^{-}5 + {}^{+}7) + (^{+}6 + {}^{-}10) = \underline{?}$

The fourth set of pages starts with an exploratory exercise that leads into many of the mathematical principles that will be discussed in other Parts. A few of the problems in this set are just for straightforward computation, but some are easily done if the student has enough awareness of the basic principles.

[Part 103] [Page 74]

A numeral has been removed from each of the sentences below. Write a numeral on your work sheet that makes the sentence true. If you cannot get the answer to a problem, go on to the next, etc. Some people can get the answer without using scratch paper. Can you?

(1) $(^{+}50 + ^{-}20) + ^{-}7 =$ ___?___ $+ ^{+}30$

(2) ___?___ $+ (^{-}61 + ^{-}28) = ^{-}89 + ^{+}10$

(3) $(^{+}5 + ^{-}75) + ^{-}25 =$ ___?___ $+ ^{-}100$

(4) ___?___ $+ (^{+}32 + ^{-}12) = ^{+}28$

(5) (___?___ $+ ^{-}10) + (^{-}10 + ^{-}10) = ^{+}18 + ^{-}30$

(6) $(^{-}372 + ^{+}1) + (^{+}50 + ^{+}30) = ^{-}372 +$ ___?___

(7) $(^{-}109 + ^{+}3) +$ (___?___ $+ ^{-}1) = ^{-}110 + ^{+}10$

(8) $(^{+}76 +$ ___?___ $) + ^{-}76 = ^{+}38$

(9) $(^{-}158 + ^{+}287) + (^{-}287 + ^{+}160) =$ ___?___

(10) $(^{-}83 + ^{+}76) + (^{-}42 + ^{+}50) =$ ___?___

[Hint: $(^{+}83 + ^{-}76) + (^{+}42 + ^{-}50) = ^{-}1$]

[Part 103] [Page 75]

Check your answers.

(1) $^{-}7$ (2) $^{+}10$ (3) $^{+}5$ (4) $^{+}8$ (5) $^{+}18$

(6) $^{+}81$ (7) $^{+}7$ (8) $^{+}38$ (9) $^{+}2$ (10) $^{+}1$

Record your results.

* * *

Refer to the road map to fill the blanks on your work sheet. On the last column, give the measure of the direct trip using the indicated-sum form so that the addends will be displayed.

	First Trip	Second Trip	Direct Trip	Sum
Example:	R to J,	J to M	R to M	$^{-}4 + ^{-}1$
(1)	R to J,	___?___ to T	R to T	___?___ $+ ^{+}1$
(2)	R to J,	J to A	R to ___?___	$^{-}4 +$ ___?___
(3)	R to J,	J to K	___?___ to ___?___	___?___ + ___?___
(4)	R to J,	J to R	___?___ to ___?___	___?___ + ___?___

[Part 103] [Page 76]

Check your answers.

	First Trip	Second Trip	Direct Trip	Sum
(1)	R to J,	J to T	R to T	$^{-}4 + ^{+}1$
(2)	R to J,	J to A	R to A	$^{-}4 + ^{+}2$
(3)	R to J,	J to K	R to K	$^{-}4 + ^{+}3$
(4)	R to J,	J to R	R to R	$^{-}4 + ^{+}4$

Record your results.

* * *

Refer to the road map to fill the blanks on your work sheet. In the last column, give the measure of the direct trip.

	First Trip	Second Trip	Direct Trip	Sum
(1)	M to J,	J to ___?___ . . .	M to ___?___	$^{+}4$
(2)	M to T,	___?___ to K . . .	___?___ to ___?___	___?___
(3)	M to A,	___?___ to ___?___ . . .	___?___ to ___?___	$^{+}4$
(4)	M to K,	___?___ to ___?___ . . .	___?___ to K	___?___

Check your answers.

	First Trip	Second Trip	Direct Trip	Sum
(1)	M to J,	J to K	. . . M to K	$^{+}4$
(2)	M to T,	T to X	. . . M to K	$^{+}4$
(3)	M to A,	A to K	. . . M to K	$^{+}4$
(4)	M to K,	K to K	. . . M to K	$^{+}4$

Record your results.

* * *

The road map is not shown for these problems:

Fill the blanks on your work sheet.

			measure
(1)	First trip	P to V	$^{+}3$
	Second trip	? to ?	●
	Direct trip	P to V	?
(2)	First trip	F to R	$^{-}5$
	Second trip	R to F	?
	Direct trip	? to ?	●

We do not want you to see what is written here.

Check your answers.

			measure
(1)	First trip	P to V	$^{+}3$
	Second trip	V to V	●
	Direct trip	P to V	$^{+}3$
(2)	First trip	F to R	$^{-}5$
	Second trip	R to F	$^{+}5$
	Direct trip	F to F	●

Record your results.

* * *

Fill the blanks so that the operation is performed correctly for each problem.

(1) $(^{-}7, ^{+}6)$ → ADDITION → ?

(2) $(^{-}3, ?)$ → ADDITION → $^{-}8$

(3) $(^{-}2, ^{+}2)$ → ADDITION → (●, $^{+}5$) → ADDITION → ?

(4) $(^{+}4, ?)$ → ADDITION → (●, $^{-}3$) → ADDITION → $^{-}3$

Check your answers.

(1) $^{-}7 + ^{+}6 = ^{-}1$ (2) $^{-}3 + ^{-}5 = ^{-}8$

(3) $(^{-}2 + ^{+}2) + ^{+}5 = ^{+}5$ (4) $(^{+}4 + ^{-}4) + ^{-}3 = ^{-}3$

Record your results.

NO ERRORS . . . . . go to the '☞' on PAGE 81.
ONE OR MORE ERRORS . . . continue to read on this page.

* * *

Tell the sum for each pair of addends:

	First Addend	Second Addend	Sum
Example:	$^{+}3$	$^{-}2 + ^{-}4$	?
Solution:	$^{+}3$	$^{-}6$	$^{-}3$
(1)	$^{-}2$	$^{+}3$	?
(2)	$^{+}4$	$^{+}1 + ^{-}3$	?
(3)	$^{-}1$	$^{+}7 + ^{-}3$	?
(4)	$^{+}4$	$^{-}3 + ^{+}3$	?
(5)	$^{-}5 + ^{+}5$	$^{+}5$	?
(6)	$^{-}4 + ^{+}10$	$^{+}7 + ^{-}7$	?

PART 104

Content

Part 104 teaches about multiplication of real numbers. The basic methods employed in teaching the topic in Part 104 are the same as the ones used in Unit 1, UICSM High School Mathematics. The use of the Pump-Tank-Film model for multiplication of real numbers has proved effective in classroom use, and its description in the programed edition together with pictures makes this model still more meaningful. This part also uses the "function machines" to help the student see through the hard problems.

Approach

Part 104 develops the notion of multiplication of real numbers very gradually, building slowly on the students intuitive feeling for the analogy to the mathematical model which he is presented. A short "zebra" section occurs as a remedial part on multiplication of fractions. The part finishes with several pages of problems of multiplication of real numbers without particular reference to the model. The student is left on his own to devise his shortcuts or think back to the model as he feels necessary.

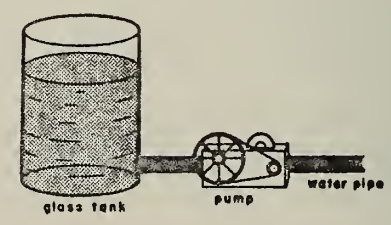
Sample pages from Part 104

The following sample pages should give the reader some insight into the advantage of this approach to multiplication of real numbers.

The first four pages show how the pump and tank are used to get across the basic notion that pumping water for a period of time can be represented with real numbers.

[Part 104] [Page 1]

A PUMP - AND - TANK PROBLEM



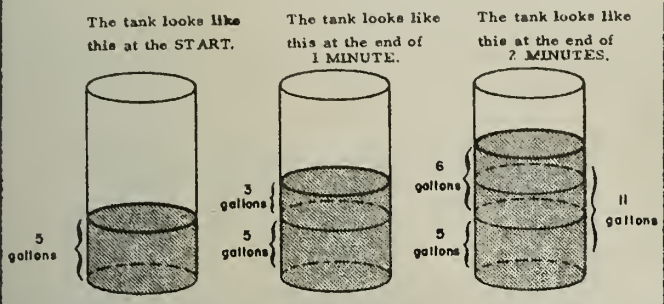
Suppose that a pump can pump water into a tank at the rate of 3 gallons per minute. If there are 5 gallons of water in the tank at the start, how many gallons are there in the tank at the end of 2 minutes of pumping?

Write your answer in the blank on your work sheet.

[Part 104] [Page 2]

Check your answer.

There are 11 gallons of water in the tank after 2 minutes of pumping. [We can assume, of course, that the tank can hold this much water.]



Record your result.

* * *

Suppose that the pump can be made to pump water into the tank at the rate of 4 gallons per minute. If there are 10 gallons in the tank at the start, how many gallons more are there in the tank at the end of 5 minutes than there were at the start?

Write the answer on your work sheet.

[Part 104] [Page 13]

Solve these problems. [The first has been solved for you as a sample. Write the answers to the others on your work sheet.]

(a) Pump . . . . . 9 gallons per minute out of tank
Time . . . . . 2 minutes of pumping
Change in water-volume . . . . . decrease of 18 gallons

(1) Pump . . . . . 7 gallons per minute into tank
Time . . . . . 4 minutes of pumping
Change in water-volume . . . . . ? of ? gallons

(2) Pump . . . . . 7 gallons per minute out of tank
Time . . . . . 4 minutes of pumping
Change . . . . . ? of ? gallons

(3) Pump . . . . . 11 gallons per minute out of tank
Time . . . . . 3 minutes of pumping
Change . . . . . ? of ? gallons

(4) Pump . . . . . 6 gallons per minute into tank
Time . . . . . $2\frac{1}{3}$ minutes of pumping
Change . . . . . ? of ? gallons

[Part 104] [Page 14]

Check your answers to the problems on Page 13.

(1) Pump . . . . . 7 gallons per minute into tank
Time . . . . . 4 minutes of pumping
Change in water-volume . . . . . increase of 28 gallons

(2) Pump . . . . . 7 gallons per minute out of tank
Time . . . . . 4 minutes of pumping
Change . . . . . decrease of 28 gallons

(3) Pump . . . . . 11 gallons per minute out of tank
Time . . . . . 3 minutes of pumping
Change . . . . . decrease of 33 gallons

(4) Pump . . . . . 6 gallons per minute into tank
Time . . . . . $2\frac{1}{3}$ minutes of pumping
Change . . . . . increase of 14 gallons

Record your results.

* * *

Here are two more pump-tank problems. Write your answers on your work sheet.

(1) A pump can put water into a tank at the rate of 5 gallons per minute. If the pump is allowed to work for 0 minutes, then the change in the water-volume of the tank is ? gallons.

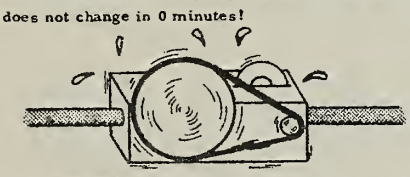
(2) [Same problem as above, but make it 'out of' instead of 'into'.]

[Part 104] [Page 15]

Check your answers.

If you answered that the change in water-volume is 0 gallons for problems (1) and (2), you are correct.

No matter how fast the pump can work, the water-volume does not change in 0 minutes!



To say the following:

Pump . . . . . 5 gallons per minute into [or: out of] the tank
Time . . . . . 0 minutes of pumping
Change in water-volume . . . 0 gallons

corresponds to saying:

$$5 \times 0 = 0$$

Record your results.

* * *

Here is a peculiar pump-tank problem:

A pump is pumping 0 gallons of water per minute. [This isn't much of a pump!] If the pump is allowed to work for 3 minutes, what change takes place in the water-volume?

Write your answer on the work sheet.

The next seven pages develop the model still further, and encourage the student to use his imagination to complete the model.

[Part 104] [Page 16]

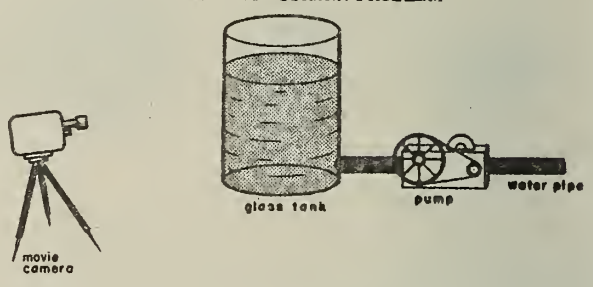
Check your answer and record your result.

If you thought that the water-volume stayed the same because this was a good-for-nothing pump, you were right.

A 0-gallon-per minute pump working for 3 minutes will pump a total of 0 [0 × 3 = 0] gallons of water. Therefore, the water-volume stays the same. So we say that there was a change of 0 gallons in the water-volume.

* * *

A PUMP-TANK-MOVIE-SCREEN PROBLEM



Suppose that the pump is putting water into the tank at a rate of 5 gallons per minute, and that the pump works for 3 minutes.

Suppose, further, that someone is taking a motion picture of the tank during the entire process. [Since the tank has glass walls, the camera can "see" what is happening.]

What change in the water-volume does the camera see?

Write your answer on your work sheet.

[Part 104] [Page 17]

Check your answer.

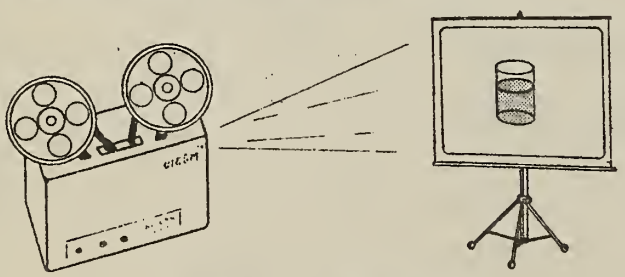
Since the pump pnts 5 gallons of water into the tank each minute, and since it works for 3 minutes, the water-volume *increases* by *15* gallons. This is what the motion picture camera sees.

Record your result.

* * *

Now, imagine that the film has been developed and a print has been made.

Someone takes the finished film into a darkened room which has a movie projector and a screen. He puts the film into the projector, and turns it on. Then he observes the movie on the screen.



What change in the water-volume does he observe on the screen?

Write your answer on your work sheet.

[Part 104] [Page 18]

Check your answer.

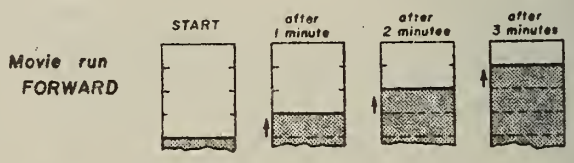
Did you say that what the man observed on the screen was an increase of 15 gallons?

If you did, you're wrong. And, it's our fault, too!

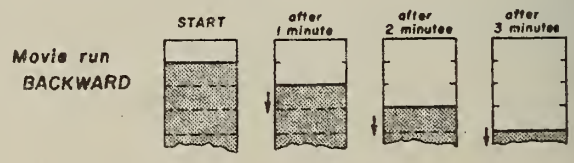
We forgot to tell you one very important thing. He put the film into the projector in such a way that the movie would run backward. So, when he turned on the projector, what he actually observed on the screen was a decrease of 15 gallons!

[Have you ever watched a comedy film in which a man seems to dive up out of the water and land on the diving board?]

Here is what the camera saw as the water was being pumped into the tank:



But, here is what the man with the projector saw on the screen when he ran the movie backward:

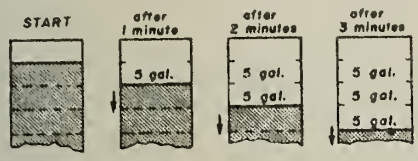


So, although the water was being pumped into the tank, a person looking at the movie run backward would see a decrease in the water-volume.

Here is the start of another pump-movie-screen problem:

> Water is being pumped out of a tank at the rate of 5 gallons per minute. A 3-minute movie is made of the tank while the pump is working.

This is what the camera sees as the water is being pumped out of the tank:



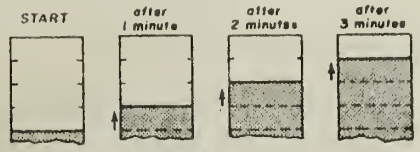
The film is developed and a print is made. The finished movie is put into a projector, and the projector is turned on.

(1) Draw the four water-levels [on your work sheet] to show what you would see on the screen if the movie were run backward.

(2) On your work sheet tell the change in water-volume you would see on the screen if the movie were run backward.

Check your answers.

(1) Here is what you would see on the screen.



Movie run BACKWARD

(2) Since the pump is an out-of-pump, the last thing the movie camera sees is the lowest level of water. And, since the movie is projected backward, this is the first thing you see on the screen. So, you observe the water-level go up.

In other words, the change you see on the screen is an increase in the water-volume.

What is the size of the change? 5 gallons per minute for 3 minutes amounts to 15 gallons.

So the observed change is *an increase of 15 gallons*.

Record your results.

* * *

Here is another pump-movie-screen problem. Fill the blank on your work sheet.

Pump. . . . . 5 gallons per minute into tank
Movie . . . . . 3 minutes run forward
Change on screen . . . ___?___ of 15 gallons

Practice with these pump-movie-screen problems. Two samples are worked for you. Write answers to the others on your work sheet.

(a) 8 gal. per min. out of tank — 5 minutes of movie backward
[*decrease*] → [*looks opposite*]
Change on screen . . . . . *increase* of *40* gallons

(b) 2 gal. per min. into tank — 7 minutes of movie forward
[*increase*] → [*looks same*]
Change on screen . . . . . *increase* of *14* gallons

(1) 6 gal. per min. into tank — 3 minutes of movie forward
Change on screen . . . . . ___?___ of ___?___ gallons

(2) 5 gal. per min. into tank — 6 minutes of movie backward
Change on screen . . . . . ___?___ of ___?___ gallons

(3) 4 gal. per min. out of tank — 5 minutes of movie forward
Change on screen . . . . . ___?___ of ___?___ gallons

(4) 5 gal. per min. into tank — 4 minutes of movie backward
Change on screen . . . . . ___?___ of ___?___ gallons

(5) 5 gal. per min. out of tank — 4 minutes of movie backward
Change on screen . . . . . ___?___ of ___?___ gallons

Here are the answers to the problems on Page 28.

	PUMP	MOVIE		
(1)	6 gals. per min. into [*increase*]	3 minutes forward [*looks same*]	[6 × 3 = *18*]	Change on screen . . . . . *increase* of *18* gallons
(2)	5 gals. per min. into [*increase*]	6 minutes backward [*looks opposite*]	[5 × 6 = 30]	Change on screen . . . . . *decrease* of *30* gallons
(3)	4 gals. per min. out of [*decrease*]	5 minutes forward [*looks same*]	[4 × 5 = *20*]	Change on screen . . . . . *decrease* of *20* gallons
(4)	5 gals. per min. into [*increase*]	4 minutes backward [*looks opposite*]	[5 × 4 = *20*]	Change on screen . . . . . *decrease* of *20* gallons
(5)	5 gals. per min. out of [*decrease*]	4 minutes backward [*looks opposite*]	[5 × 4 = *20*]	Change on screen . . . . . *increase* of *20* gallons

Record your results.

The text then advances the student to the point at which he is using

'+' for pumping water into the tank and

'−' for pumping it out;

'+' for running film forward and

'−' for running it backward;

'+' for increase in water level on screen and

'−' for decrease in water level on screen.

The transition is then made to multiplication of real numbers. The student may still refer to the model whenever he need it.

[Part 104] [Page 38]

You are correct.

$(^{+}6, {}^{-}2)$.....$^{-}12$

does correspond to:

6 gallons per minute into tank
2 minutes movie backward
screen shows decrease of 12 gallons

Record your result.

* * *

Here are some more pump-movie-screen problems.
Write the answers on your work sheet.

(1) $(^{+}4, {}^{-}3)$....._?_ — Pump.....$^{+}4$ gallons per minute; Movie.....$^{-}3$ minutes; Change on screen....._?_ gallons

(2) $(^{-}5, {}^{+}7)$....._?_ — Pump.....$^{-}5$ gallons per minute; Movie.....$^{+}7$ minutes; Change on screen....._?_ gallons

(3) $(^{+}2, {}^{+}9)$....._?_ — Pump.....$^{+}2$ gallons per minute; Movie.....$^{+}9$ minutes; Change on screen....._?_ gallons

(4) $(^{-}8, {}^{-}1)$....._?_ — Pump.....$^{-}8$ gallons per minuts; Movie.....$^{-}1$ minutes; Change on screen....._?_ gallons

[Part 104] [Page 40]

Here are answers to the problems on Page 38, and some hints in case you missed any.

(1) $(^{+}4, {}^{-}3)$.....$^{-}12$ — Pump.....$^{+}4$ gal. per minute [increase]; Movie.....$^{-}3$ minutes [backward]; Change on screen.....$^{-}12$ gallons [decrease]

(2) $(^{-}5, {}^{+}7)$.....$^{-}35$ — Pump.....$^{-}5$ gal. per minute [decrease]; Movie.....$^{+}7$ minutes [forward]; Change on screen.....$^{-}35$ gallons [decrease]

(3) $(^{+}2, {}^{+}9)$.....$^{+}18$ — Pump.....$^{+}2$ gal. per minute [increase]; Movie.....$^{+}9$ minutes [forward]; Change on screen.....$^{+}18$ gallons [increase]

(4) $(^{-}8, {}^{-}1)$.....$^{+}8$ — Pump.....$^{-}8$ gal. per minute [decrease]; Movie.....$^{-}1$ minute [backward]; Change on screen.....$^{+}8$ gallons [increase]

Record your results.

[Part 104] [Page 45]

Here are the answers to the problems on Page 44.

(1) $(^{-}10, {}^{-}3)$.....$^{+}30$ (2) $(^{+}10, {}^{+}3)$.....$^{+}30$
(3) $(^{-}10, {}^{+}3)$.....$^{-}30$ (4) $(^{+}1, {}^{-}1)$.....$^{-}1$
(5) $(^{-}1, {}^{+}1)$.....$^{-}1$ (6) $(^{-}1, {}^{-}1)$.....$^{+}1$

Record your results.

* * *

Here is a big list of pump-movie-screen examples. Exactly one example is wrong. When you find it, write a correct version on your work sheet.

$(^{+}7, {}^{-}1)$.....$^{-}7$ $(^{-}2, {}^{+}1)$.....$^{-}2$ $(^{-}4, {}^{-}4)$.....$^{+}16$

$(^{-}5, {}^{+}5)$.....$^{-}25$ $(0, {}^{+}6)$.....0 $(^{+}8, {}^{-}2)$.....$^{-}16$

$(^{+}100, {}^{+}3)$.....$^{+}300$ $(^{-}51, {}^{+}2)$.....$^{-}102$

$(^{-}8, {}^{+}\frac{1}{2})$.....$^{-}4$ $(^{+}8, {}^{-}\frac{1}{2})$.....$^{-}4$ $(^{+}\frac{1}{3}, {}^{-}\frac{1}{2})$.....$^{-}\frac{1}{6}$

$(^{-}\frac{1}{3}, {}^{-}\frac{1}{2})$.....$^{+}\frac{1}{6}$ $(^{+}832, {}^{-}68)$.....$^{-}56576$ $(^{-}3, {}^{+}1)$.....$^{-}3$

$(^{-}68, {}^{+}832)$.....$^{+}56576$ $(^{-}8, 0)$.....0 $(0, 0)$.....0

$(^{-}5, {}^{+}2)$.....$^{-}10$ $(^{-}4.3, {}^{-}2)$.....$^{+}8.6$ $(^{-}3, {}^{-}1)$.....$^{+}3$

The next section is a presentation of Parts 105, 106, and 107. The following exerpts are taken from the tests for these parts to illustrate some of their objectives.

Circle the letter corresponding to the correct choice.

A B C D The two sets of numbers listed in the table at the right "act alike" with respect to

A) addition
B) multiplication
C) both of these
D) neither of these

2	8
3	27
4	64
5	125
6	216
8	512

Fill the blanks.

The numbers of Set A [in the table at the right] together with the operation _____ act like the numbers of Set B together with the operation _____.

A	B.
1	0
10	1
100	2
1, 000	3
10, 000	4
100, 000	5
1, 000, 000	6

Unabbreviate Fully the following expressions by writing grouping symbols directly in them where necessary. [It is not necessary to include outermost grouping symbols.]

$$[72 + (5\tfrac{1}{2} - \tfrac{1}{2} \times 10)] - 6 + 11$$

$$36 - (26 + 8 \div 3) + (17 - 11 + 9) + 3$$

. . .

Simplify.

$$7 + 2 \times 3 + 4$$

$$105 - \{17 - 8 + (35 - 21 \div 7)\} \times 2$$

. . .

PART 105

At this point, the students have learned how to add and multiply pairs of real numbers. These operations were taught with the aid of physical devices [Trips on a road, and Pump-tank-movie] but the exercises were devised so that the students would discover short cuts for themselves. They are aware that there is some "connection" between the numbers of arithmetic and the real numbers. In fact some of the students may have been told that the numbers of arithmetic <u>are</u> the nonnegative real numbers. Part 105 has been written as an introduction to isomorphisms. The students should discover that the nonnegative real numbers are isomorphic to ("act like") the numbers of arithmetic with respect to addition and multiplication; but, "acting alike" with respect to addition is no more reason to say they are the same than it is to say

$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$

and

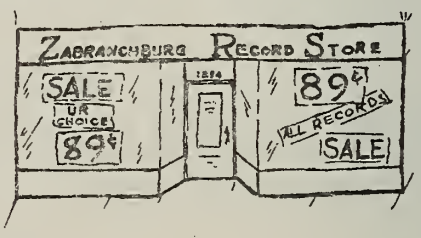
$\{89, 178, 267, 356, 445, 534, 623, 712, 801, 890, 979, 1068, 1157, 1246, 1335, 1424, 1513, 1602, 1691, 1780\}$

are the same.

Part 105 starts with a "story problem" which leads the student into using a table to compute sums of multiples of 89.

[Part 105] [Page 1]

Walt Barnes was hired one Saturday by the owner of a small record store in his neighborhood. Walt was to work as a clerk while the store was having a "clean-out" sale on all popular records.



A customer would make his choice of records and then Walt would note the amount of the purchase on a sales slip. The customer would take the sales slip to the cashier while Walt wrapped the purchase. The customer then took his receipt to Walt and got his wrapped purchase. [To make things simpler, we will not consider sales tax or any other taxes.]

A customer came to Walt with 5 records. Since, during this sale, each record was priced at 89¢, the amount of the customer's purchase was <u>?</u>.

Write the answer to this question on your work sheet.

[Part 105] [Page 2]

Check your answer.

The amount of the customer's purchase was $4.45.

* * *

Answer the following questions on your work sheet.

Three more customers came to Walt during the "89¢ sale".

(a) The first customer had 3 records; the amount of the purchase was ___?___.

(b) The second customer had 7 records; the amount of his purchase was ___?___.

(c) The third customer had 10 records; the amount of his purchase was ___?___.

[Part 105] [Page 3]

Check your answers.

(a) 3 records cost $2.67.

(b) 7 records cost $6.23.

(c) 10 records cost $8.90.

* * *

In order to make his job easier, Walt started making a table. The column labeled 'R' listed the number of records and the column labeled 'A' listed the amount of the purchase in cents.

Complete Walt's table.

Write your answers in the space provided on your work sheet.

R		A
1	. . .	89
2	. . .	?
3	. . .	267
4	. . .	?
5	. . .	445
6	. . .	?
7	. . .	623
8	. . .	?
9	. . .	?
10	. . .	890

[Part 105] [Page 4]

Check your answers.

	R		A
	1	. . .	89
(a)	2	. . .	178
	3	. . .	267
(b)	4	. . .	356
	5	. . .	445
(c)	6	. . .	534
	7	. . .	623
(d)	8	. . .	712
(e)	9	. . .	801
	10	. . .	890

Record your results in the result-check column on your work sheet.

* * *

Complete the following to make true sentences by filling the blanks on your work sheet.

(a) 3 records cost ___?___.

(b) ___?___ records cost $8.01.

(c) 4 records cost _____.

(d) ___?___ records cost $7.12.

[Part 105] [Page 5]

Check your answers.

(a) 3 records cost $2.67.

(b) 9 records cost $8.01.

(c) 4 records cost $3.56.

(d) 8 records cost $7.12.

Record your results in the result-check column on your work sheet.

R		A
1	. . .	89
2	. . .	178
3	. . .	267
4	. . .	356
5	. . .	445
6	. . .	534
7	. . .	623
8	. . .	712
9	. . .	801
10	. . .	890
⋮		⋮

* * *

A customer came to Walt with a selection of records, for which Walt made out a sales slip for $2.67. The customer then picked out some more records worth $5.34.

Answer the following questions on your work sheet.

(a) The '$2.67' on the sales slip indicated that the customer had first picked out ___?___ records.

(b) The customer's second selection of ___?___ records cost $5.34.

(c) The total number of records the customer purchased was ___?___.

(d) The total amount of the customer's purchase was $___?___.

The reader should note that the original problem (Finding the total cost of a certain number of 89¢ records) has been discarded and that the table is being used just to compute sums of multiples of 89.

Note that the table with which we have been working has been completed to the twentieth entry.

[Part 105] [Page 10]

Check your answers.

	instrumental	vocal	total
(a)	$ 6.23	$ 9.79	$16.02
(b)	$10.68	$ 5.34	$16.02
(c)	$ 3.56	$12.46	$16.02

Record your results.

* * *

Write the answers to these addition problems on your work sheet.

(1) $\begin{array}{r} 267 \\ +\ 890 \\ \hline ? \end{array}$ (2) $\begin{array}{r} 623 \\ +\ 1068 \\ \hline ? \end{array}$

(3) 267 + 267 = ?

(4) 890 + 267 = ?

R	A
1	89
2	178
3	267
4	356
5	445
6	534
7	623
8	712
9	801
10	890
11	979
12	1068
13	1157
14	1246
15	1335
16	1424
17	1513
18	1602
19	1691
20	1780

Table 1

The students have been told to read:

$\longleftrightarrow$

as they would read:

corresponds with

The problems on page 16 are to give the students an understanding of how the table "works".

[Part 105] [Page 16]

Check your answers.

[set R]		[set A]
3	$\longleftrightarrow$	267
6	$\longleftrightarrow$	534
9	$\longleftrightarrow$	801

3 + 6 = 9 801 = 267 + 534

Record your results.

* * *

Fill the blanks on your work sheet.

(1)

[set R]		[set A]
4	$\longleftrightarrow$	?
7	$\longleftrightarrow$	?
11	$\longleftrightarrow$	?

4 + 7 = ? ? = 356 + 623

(2)

[set R]		[set A]
(15 + 2)	$\longleftrightarrow$	(1335 + ?)
17	$\longleftrightarrow$	(1335 + ?)
?	$\longleftrightarrow$	1513

(3)

[set A]		[set R]
(623 + 979)	$\longleftrightarrow$	(? + 11)
(623 + ?)	$\longleftrightarrow$	18

(4) 623 + 979 = ?

R	A
1	89
2	178
3	267
4	356
5	445
6	534
7	623
8	712
9	801
10	890
11	979
12	1068
13	1157
14	1246
15	1335
16	1424
17	1513
18	1602
19	1691
20	1780

Table 1

The problems on pages 25, 26, and 27 point out the limitations of Table 1.

[Part 105] [Page 25]

Check your answers and record your results.

(1) The sum (1691 + 89) in set A is listed in the table and the corresponding sum ($\underset{(a)}{19}$ + 1) in set R is listed in the table.

So, using the table, we can simplify:

1691 + 89 [set A]

by simplifying:

$\underset{(a)}{19}$ + 1 [set R]

(2) The sum (1780 + 89) in set A is NOT LISTED in the table and the corresponding sum ($\underset{(a)}{20}$ + 1) in set R is NOT LISTED in the table.

So, we $\underset{(b)}{\text{cannot}}$ use this table AS IT STANDS to simplify:

1780 + 89 [set A]

by simplifying:

20 + $\underset{(a)}{1}$ [set R]

* * *

R	A
1	89
2	178
3	267
4	356
5	445
6	534
7	623
8	712
9	801
10	890
11	979
12	1068
13	1157
14	1246
15	1335
16	1424
17	1513
18	1602
19	1691
20	1780

Table 1

Complete the following by filling the proper blanks on your work sheet.

(1) We can use this table as it stands to simplify '1691 + 89' [set A] by simplifying '$\underset{(a)}{?}$ + 1' [set R] because the sum (1691 + 89) in set $\underset{(a)}{?}$ is listed in the table and the corresponding sum (19 + $\underset{(a)}{?}$) in set R is listed in the table.

(2) We CANNOT use table as it stands to simplify '1513 + 356' [set A] by simplifying '$\underset{(a)}{?}$ + 4' [set R] because the sum (1513 + 356) in set $\underset{(a)}{?}$ is NOT LISTED in the table and the corresponding sum (17 + $\underset{(a)}{?}$) in set R is NOT LISTED in the table.

[Part 105] [Page 26]

Check your answers.

(1) We can use this table as it stands to simplify '1691 + 89' [set A] by simplifying '$\underset{(a)}{19}$ + 1' [set R] because the sum (1691 + 89) in set $\underset{(a)}{A}$ is listed in the table and the corresponding sum (19 + $\underset{(a)}{1}$) in set R is listed in the table.

(2) We cannot use the table as it stands to simplify '1513 + 356' [set A] by simplifying '$\underset{(a)}{17}$ + 4' [set R] because the sum (1513 + 356) in set $\underset{(a)}{A}$ is not listed in the table and the corresponding sum (17 + $\underset{(a)}{4}$) in set R is not listed in the table.

Record your results.

* * *

R	A
1	89
2	178
3	267
4	356
5	445
6	534
7	623
8	712
9	801
10	890
11	979
12	1068
13	1157
14	1246
15	1335
16	1424
17	1513
18	1602
19	1691
20	1780

Table 1

Since each sum of set A listed in the table corresponds with a sum of set R listed in the table, we can use this table for some addition problems.

Complete the following:

We can say that set A and set R

"act alike"

with respect to ___?___.

[Part 105] [Page 27]

Check your answer.

Set A and set R

"act alike"

with respect to *addition*.

Record your results.

* * *

As long as we have this table to look at, try this MULTIPLICATION problem:

Be sure both sentences are TRUE.

[set A] 178 × 534 = ___?___

[set R] ___?___ × ___?___ = ___?___

Does this correspond with this? ___?___

R	A
1	89
2	178
3	267
4	356
5	445
6	534
7	623
8	712
9	801
10	890
11	979
12	1068
13	1157
14	1246
15	1335
16	1424
17	1513
18	1602
19	1691
20	1780

Table 1

The table which appears on page 52a is the fifth one that the students have investigated. It is of a different form than the others and the students are asked to determine the operations for which it "works".

Notice that page 52b is the start of a branch sequence in the booklet. The student is directed to one of four sequences, depending upon his answer to the question at the top of the page.

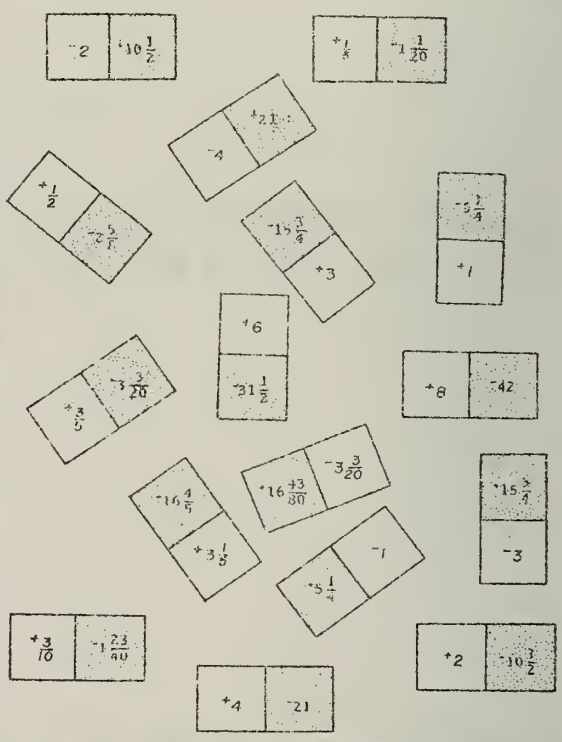
Answer	Sequence of pages
(a)	52b → 53 → 57
(b)	52b → 55 → 57
(c)	52b → 56 → 57
(d)	52b → 54 → 57

[Page 52a] [Part 105]

Here is another table which can be used as you used the other tables.

Figure out how to use it and see if it works for multiplication or addition,

then go to facing page.



[Part 105] [Page 52b]

Complete by filling the blank on your work sheet.

The table on the facing page [52a] works for _____?_____ problems.

(a) some multiplication, but no addition

(b) some addition, but no multiplication

(c) both multiplication and addition

(d) neither multiplication nor addition

If your answer is:

(a), turn to PAGE 53.

(b), turn to PAGE 55;

(c), turn to PAGE 56.

(d), turn to PAGE 54.

The tables referred to on pages 53b, 54 b, 55b, and 56b are copies of the table above.

[Part 105] [Page 53b]

You said that the table on the facing page works for some multiplication but no addition problems.

You are wrong.

* * *

Let's call the sets listed in the table

'K' for 'darK'

and 'L' for 'Light'.

Then, we can think of the table being re-arranged to look like this:

	$^-4$	$^-3\frac{3}{20}$	$^-3$	$^-2$	$^-1$	$^+\frac{1}{5}$	$^+\frac{3}{10}$	$^+\frac{1}{2}$	$^+\frac{3}{8}$	$^+1$	$^+2$	$^+3$	$^+3\frac{1}{8}$	$^+4$	$^+6$	$^+8$
	$^+21$	$^+16\frac{43}{80}$	$^+15\frac{3}{4}$	$^+10\frac{1}{2}$	$^+5\frac{1}{4}$	$^-1\frac{1}{20}$	$^-1\frac{23}{40}$	$^-2\frac{5}{8}$	$^-3\frac{3}{20}$	$^-3\frac{1}{4}$	$^-10\frac{1}{2}$	$^-15\frac{3}{4}$	$^-16\frac{4}{5}$	$^-21$	$^-31\frac{1}{4}$	$^-42$

To show how the table works, complete the following examples by filling the blanks on your work sheet.

Be sure each sentence is true.

(1) [set ?] $^-15\frac{3}{4}$ + ? = ?

Addition

[set ?] $^+3$ + $^+\frac{1}{5}$ = ?

Do these correspond? ?

(2) [set ?] $^-15\frac{3}{4}$ × ? = ?

Multiplication

[set ?] $^+3$ × $^+\frac{1}{5}$ = ?

Do these correspond? ?

[Part 105] [Page 54b]

You said that the table on the facing page works neither for some multiplication nor for some addition problems.

You are wrong.

* * *

Let's call the sets listed in the table

'K' for 'darK'

and 'L' for 'Light'.

Then, we can think of the table being re-arranged to look like this:

L	$^-4$	$^-3\frac{3}{20}$	$^-3$	$^-2$	$^-1$	$^+\frac{1}{5}$	$^+\frac{3}{10}$	$^+\frac{1}{2}$	$^+\frac{3}{8}$	$^+1$	$^+2$	$^+3$	$^+3\frac{1}{8}$	$^+4$	$^+6$	$^+8$
K	$^+21$	$^+16\frac{43}{80}$	$^+15\frac{3}{4}$	$^+10\frac{1}{2}$	$^+5\frac{1}{4}$	$^-1\frac{1}{20}$	$^-1\frac{23}{40}$	$^-2\frac{5}{8}$	$^-3\frac{3}{20}$	$^-3\frac{1}{4}$	$^-10\frac{1}{2}$	$^-15\frac{3}{4}$	$^-16\frac{4}{5}$	$^-21$	$^-31\frac{1}{4}$	$^-42$

To show how the table works, complete the following examples by filling the blanks on your work sheet.

Be sure each sentence is true.

(1) [set ?] $^-15\frac{3}{4}$ + ? = ?

Addition

[set ?] $^+3$ + $^+\frac{1}{5}$ = ?

Do these correspond? ?

(2) [set ?] $^-15\frac{3}{4}$ × ? = ?

Multiplication

[set ?] $^+3$ × $^+\frac{1}{5}$ = ?

Do these correspond? ?

[Part 105] [Page 55b]

You said that the table on the facing page works for

some addition,

but no multiplication problems.

You are right.

* * *

Let's call the sets listed in the table

'K' for 'darK'

and 'L' for 'Light'.

Then, we can think of the table being re-arranged to look like this:

	$^-4$	$^-3\frac{3}{20}$	$^-3$	$^-2$	$^-1$	$^+\frac{1}{5}$	$^+\frac{3}{10}$	$^+\frac{1}{2}$	$^+\frac{3}{8}$	$^+1$	$^+2$	$^+3$	$^+3\frac{1}{8}$	$^+4$	$^+6$	$^+8$
K	$^+21$	$^+16\frac{43}{80}$	$^+15\frac{3}{4}$	$^+10\frac{1}{2}$	$^+5\frac{1}{4}$	$^-1\frac{1}{20}$	$^-1\frac{23}{40}$	$^-2\frac{5}{8}$	$^-3\frac{3}{20}$	$^-3\frac{1}{4}$	$^-10\frac{1}{2}$	$^-15\frac{3}{4}$	$^-16\frac{4}{5}$	$^-21$	$^-31\frac{1}{4}$	$^-42$

To show how the table works, complete the following examples by filling the blanks on your work sheet.

Be sure each sentence is true.

(1) [set ?] $^-15\frac{3}{4}$ + ? = ?

Addition

[set ?] $^+3$ + $^+\frac{1}{5}$ = ?

Do these correspond? ?

(2) [set ?] $^-15\frac{3}{4}$ × ? = ?

Multiplication

[set ?] $^+3$ × $^+\frac{1}{5}$ = ?

Do these correspond? ?

[Part 105] [Page 56b]

You said that the table on the facing page works for

both multiplication

and addition problems.

You are wrong.

* * *

Let's call the sets listed

'K' for 'darK'

and 'L' for 'Light'.

Then we can think of the table being re-arranged to look like this:

L	$^-4$	$^-3\frac{3}{20}$	$^-3$	$^-2$	$^-1$	$^+\frac{1}{5}$	$^+\frac{3}{10}$	$^+\frac{1}{2}$	$^+\frac{3}{8}$	$^+1$	$^+2$	$^+3$	$^+3\frac{1}{8}$	$^+4$	$^+6$	$^+8$
K	$^+21$	$^+16\frac{43}{80}$	$^+15\frac{3}{4}$	$^+10\frac{1}{2}$	$^+5\frac{1}{4}$	$^-1\frac{1}{20}$	$^-1\frac{23}{40}$	$^-2\frac{5}{8}$	$^-3\frac{3}{20}$	$^-3\frac{1}{4}$	$^-10\frac{1}{2}$	$^-15\frac{3}{4}$	$^-16\frac{4}{5}$	$^-21$	$^-31\frac{1}{4}$	$^-42$

To show how the table works, complete the following examples by filling the blanks on your work sheet.

Be sure each sentence is true.

(1) [set ?] $^-15\frac{3}{4}$ + ? = ?

Addition

[set ?] $^+3$ + $^+\frac{1}{5}$ = ?

Do these correspond? ?

(2) [set ?] $^-15\frac{3}{4}$ × ? = ?

Multiplication

[set ?] $^+3$ × $^+\frac{1}{5}$ = ?

Do these correspond? ?

Table 8, which appears on pages 69-71 [see below], does not work like the previous tables. In this table the sum of a pair of numbers in set A corresponds with the product of the corresponding pair of numbers in set P.

Q	M
$^+1$	$^+3$
$^+2$	$^+6$
$^+3$	$^+9$
$^+4$	$^+12$
$^+5$	$^+15$
$^+6$	$^+18$
.	.
.	.
.	.

Table 8 brings out a subtlety for the reader that is not pressed for the beginning high school student. That is, even though we have referred to the pair of operations in previous tables by the same name, there is a difference. For example, for the table at the right we say that the table "works" for addition when we really mean that set M, under addition of positive integers restricted to set M, is isomorphic to set Q, under addition of positive integers.

[Part 105] [Page 69]

		[I]		[D]
(1)	true	$^+2$	—	$^+4$
(2)	true	($^+2 + {}^+2 + {}^+2$)	—	($^+4 + {}^+4 + {}^+4$)
(3)	true	Set I is isomorphic to set D with respect to TRIPLING.		
(4)	true	Set I and set D are isomorphic with respect to QUADRUPLING.		

Record your results.

* * *

A	0	1	2	3	4	5	6	7	8	10	11	12
P	1	2	4	8	16	32	64	128	256	1024	2048	4096

Table 8

Using Table 8, complete each of the following:

	[A]		[P]
(1)	2	—	?
(2)	?	—	16
(3)	(5 + ?)	—	2048
(4)	(4 + 1)	—	(16 + ?)

[Part 105] [Page 70]

Check your answers.

	[A]		[P]
(1)	2	—	4
(2)	4	—	16
(3)	(5 + 6)	—	2048
(4)	(4 + 1)	—	(16 + 16)

Record your results.

* * *

A	0	1	2	3	4	5	6	7	8	10	11	12
P	1	2	4	8	16	32	64	128	256	1024	2048	4096

Table 8

Complete this example to show how Table 8 can be used.

Be sure both sentences are true.

['×' or '+'?]

[set P] 4 ? 32 = ?

[set A] 2 ? 5 = 7

['×' or '+'?]

[Part 105] [Page 71]

Check your answers.

[P] 4 _X_ 32 128

[A] 2 _+_ 5 = 7

Record your results.

* * *

A	0	1	2	3	4	5	6	7	8	10	11	12
P	1	2	4	8	16	32	64	128	256	1024	2048	4096

Table 8

This is a peculiar table in that it works differently than the others you have used. In the other tables, you could solve problems in one set by solving the same kind of problem with the corresponding numbers in the other set.

Complete each of the following:

	[A]		[P]
(1)	2	⟶	?
(2)	?	⟶	32
(3)	(2 + ?)	⟶	128
(4)	7	⟶	(? × 32)
(5)	(4 + 6)	⟶	(16 × ?)
(6)	10	⟶	?
(7)	(3 + ?)	⟶	2048
(8)	8	⟶	(? × 32)

Table 12 points out the "connection" between the numbers of arithmetic and the nonpositive real numbers.

Table 13 points out the "connection" between the numbers of arithmetic and the nonnegative real numbers.

[Part 105] [Page 91]

Answers.

(1) [X] $^-3$ + $^-4$ = $^-7$ (yes)

Addition ?

[A] 3 + 4 = 7

(2) [X] $^-3$ × $^-4$ = $^+12$ (no)

Multiplication ?

[A] 3 × 4 = 12

Record your results.

* * *

Using Table 12, complete the following.

(1) The real number 0 is neither a positive number nor a ___?___ number.

(2) The real number 0 is a nonnegative real number. It is also a ___?___ real number.

(3) Each number in set A is a number of arithmetic, and each number in set X is the corresponding nonpositive ___?___ number.

(4) In the multiplication problem at the top of the page, notice that

($^-3$ × $^-4$) does NOT belong to set X

but, (3 × 4) does belong to set A.

So, we can conclude that set X and set A DO NOT act alike with respect to ___?___.

A	X
0	0
1	$^-1$
2	$^-2$
3	$^-3$
4	$^-4$
5	$^-5$
6	$^-6$
7	$^-7$
8	$^-8$
10	$^-10$
12	$^-12$
13	$^-13$
20	$^-20$
42	$^-42$

Table 12

[Part 105] [Page 93]

If we tried more examples, we could be fairly certain that set X and set A act alike with respect to (a) addition but not multiplication.

Record your result.

* * *

Using Table 13, complete the following by filling the blanks on your work sheet.

(1) [Z] $^+5$ + $^+8$ = ?

ADDITION ? ← ? (yes/no)

[A] ? + ? = ?

(2) [Z] ? × ? = ?

MULTIPLICATION ? ← ? (yes/no)

[A] 6 × 7 = ?

A	Z
0	0
1	$^+1$
2	$^+2$
3	$^+3$
4	$^+4$
5	$^+5$
6	$^+6$
7	$^+7$
8	$^+8$
10	$^+10$
12	$^+12$
13	$^+13$
20	$^+20$
42	$^+42$

Table 13

Note the method we use for pronunciation, as shown in lines 10 and 11 on page 95.

Table 14 is used to show the "connections" between the numbers of arithmetic and the real numbers.

Check your answers.

(1) The real number 0 is a nonpositive real number and also a non*negative* real number.

(2) Each number in set A is a number of arithmetic and each number in set Z is the corresponding non*negative* real number.

(3) If we checked many more examples, we could be fairly certain that set Z and set A act alike with respect to *(both) addition and multiplication*.

Mathematicians would say that sets Z and A are isomorphic with respect to addition and multiplication. Pronounce 'isomorphic' as you would 'eye-so-MORE-fic'.

Record your results.

A	Z
0	0
1	+1
2	+2
3	+3
4	+4
5	+5
6	+6
7	+7
8	+8
10	+10
12	+12
13	+13
20	+20
42	+42

* *

Sets Z and A are isomorphic or act alike with respect to addition and to multiplication. This means that you can solve any addition or multiplication problem with nonnegative real numbers just by solving the same type of problem with the ___?___ numbers of arithmetic.

When doing problems such as:

$^{+}7 + {}^{+}12 = ?$ and $^{-}7 + {}^{-}12 = ?$

most likely, you first simplify:

7 + 12

and get:

19

and, then you say that:

(1) $^{+}7 + {}^{+}12 =$ *+19*

(2) $^{-}7 + {}^{-}12 =$ *-19*

Record your results.

* * *

Complete the following:

Table 14 lists 3 sets of numbers. It "matches up" each ___?___ (1) real number and each ___?___ (2) real number with the corresponding number of arithmetic.

T	A	U
0	0	0
$^{-}\frac{1}{2}$	$\frac{1}{2}$	$^{+}\frac{1}{2}$
$^{-}1$	1	$^{+}1$
$^{-}1\frac{1}{2}$	$1\frac{1}{2}$	$^{+}1\frac{1}{2}$
$^{-}2$	2	$^{+}2$
$^{-}3$	3	$^{+}3$
$^{-}3\frac{1}{2}$	$3\frac{1}{2}$	$^{+}3\frac{1}{2}$
$^{-}4$	4	$^{+}4$
$^{-}5$	5	$^{+}5$
$^{-}6$	6	$^{+}6$
$^{-}6\frac{1}{2}$	$6\frac{1}{2}$	$^{+}6\frac{1}{2}$
$^{-}7$	7	$^{+}7$
$^{-}8$	8	$^{+}8$
$^{-}9$	9	$^{+}9$
$^{-}10$	10	$^{+}10$
$^{-}10\frac{1}{2}$	$10\frac{1}{2}$	$^{+}10\frac{1}{2}$
$^{-}11$	11	$^{+}11$
$^{-}12$	12	$^{+}12$
⋮	⋮	⋮

Table 14 matches up each *nonpositive* (1) real number and each *nonnegative* (2) real number with the corresponding number of arithmetic.

Record your results.

* * *

In Table 14, each real number corresponds with that number of arithmetic which gives the distance part of a trip measured by the real number.

Thus, in order to find sums of pairs of nonpositive or nonnegative real numbers, you could begin by thinking about the numbers of arithmetic which correspond with the real numbers, and then add those numbers of arithmetic.

Complete:

In other words--
the nonpositive real numbers and the nonnegative real numbers act like the numbers of arithmetic with respect to ___?___.

T	A	U
0	0	0
$^{-}\frac{1}{2}$	$\frac{1}{2}$	$^{+}\frac{1}{2}$
$^{-}1$	1	$^{+}1$
$^{-}1\frac{1}{2}$	$1\frac{1}{2}$	$^{+}1\frac{1}{2}$
$^{-}2$	2	$^{+}2$
$^{-}3$	3	$^{+}3$
$^{-}3\frac{1}{2}$	$3\frac{1}{2}$	$^{+}3\frac{1}{2}$
$^{-}4$	4	$^{+}4$
$^{-}5$	5	$^{+}5$
$^{-}6$	6	$^{+}6$
$^{-}6\frac{1}{2}$	$6\frac{1}{2}$	$^{+}6\frac{1}{2}$
$^{-}7$	7	$^{+}7$
$^{-}8$	8	$^{+}8$
$^{-}9$	9	$^{+}9$
$^{-}10$	10	$^{+}10$
$^{-}10\frac{1}{2}$	$10\frac{1}{2}$	$^{+}10\frac{1}{2}$
$^{-}11$	11	$^{+}11$
$^{-}12$	12	$^{+}12$
⋮	⋮	⋮

Table 14

[Part 105] [Page 99]

The nonpositive real numbers and the nonnegative real numbers act like the numbers of arithmetic with respect to *addition*.

You probably discovered this when you were looking for shortcuts in computing sums of real numbers.

Record your results.

* * *

Consider this example:

[A]	2	×	3	=	6
	↑		↑		↑ ?
[U]	$^{+}2$	×	$^{+}3$	=	? .

Complete the following:

(1) The nonnegative real numbers [set U] ___?___ (do/do not) act like the numbers of arithmetic with respect to multiplication.

(2) The nonpositive real numbers ___?___ (do/do not) act like the numbers of arithmetic with respect to multiplication.

(3) Each number in set A is a ___?___ ___?___. (3 words)

(4) Each number in set T is a ___?___ ___?___. (3 words)

(5) Each number in set U is a ___?___ ___?___. (3 words)

T	A	U
0	0	0
$^{-}\frac{1}{2}$	$\frac{1}{2}$	$^{+}\frac{1}{2}$
$^{-}1$	1	$^{+}1$
$^{-}1\frac{1}{2}$	$1\frac{1}{2}$	$^{+}1\frac{1}{2}$
$^{-}2$	2	$^{+}2$
$^{-}3$	3	$^{+}3$
$^{-}3\frac{1}{2}$	$3\frac{1}{2}$	$^{+}3\frac{1}{2}$
$^{-}4$	4	$^{+}4$
$^{-}5$	5	$^{+}5$
$^{-}6$	6	$^{+}6$
$^{-}6\frac{1}{2}$	$6\frac{1}{2}$	$^{+}6\frac{1}{2}$
$^{-}7$	7	$^{+}7$
$^{-}8$	8	$^{+}8$
$^{-}9$	9	$^{+}9$
$^{-}10$	10	$^{+}10$
$^{-}10\frac{1}{2}$	$10\frac{1}{2}$	$^{+}10\frac{1}{2}$
$^{-}11$	11	$^{+}11$
$^{-}12$	12	$^{+}12$
⋮	⋮	⋮

Table 14

[Part 105] [Page 100]

Check your answers.

[Listed in reverse order.]

(5) Each number in set U is a *nonnegative real number*.

(4) Each number in set T is a *nonpositive real number*.

(3) Each number in set A is a *number of arithmetic*.

(2) The nonpositive real numbers *do not* act like the numbers of arithmetic with respect to multiplication.

(1) The nonnegative real numbers [set U] *do* act like the numbers of arithmetic with respect to multiplication.

Record your results.

* * *

Complete the following:

The nonnegative real numbers act like the numbers of arithmetic with respect to both ___?___ (1) and ___?___ (2).

T	A	U
0	0	0
$^{-}\frac{1}{2}$	$\frac{1}{2}$	$^{+}\frac{1}{2}$
$^{-}1$	1	$^{+}1$
$^{-}1\frac{1}{2}$	$1\frac{1}{2}$	$^{+}1\frac{1}{2}$
$^{-}2$	2	$^{+}2$
$^{-}3$	3	$^{+}3$
$^{-}3\frac{1}{2}$	$3\frac{1}{2}$	$^{+}3\frac{1}{2}$
$^{-}4$	4	$^{+}4$
$^{-}5$	5	$^{+}5$
$^{-}6$	6	$^{+}6$
$^{-}6\frac{1}{2}$	$6\frac{1}{2}$	$^{+}6\frac{1}{2}$
$^{-}7$	7	$^{+}7$
$^{-}8$	8	$^{+}8$
$^{-}9$	9	$^{+}9$
$^{-}10$	10	$^{+}10$
$^{-}10\frac{1}{2}$	$10\frac{1}{2}$	$^{+}10\frac{1}{2}$
$^{-}11$	11	$^{+}11$
$^{-}12$	12	$^{+}12$
⋮	⋮	⋮

Table 14

[Part 105] [Page 101]

The nonnegative real numbers act like the numbers of arithmetic with respect to both *addition* (1) and *multiplication* (2).

Record your results.

* * *

SHORT NAMES FOR NONNEGATIVE NUMBERS

The fact expressed in the box above makes it possible for us to use NAMES of numbers of arithmetic when we wish to talk about nonnegative numbers. And, we shall never make a mistake by doing this. Either we can tell from the context which numbers are intended, or else it won't make any difference.

If,

we wanted to

state the fact that

(A) $^{+}9 + {}^{-}3 = {}^{+}6$

and wrote:

(B) $9 + {}^{-}3 = 6$

then, for this sentence

to make sense

we would have to agree

that:

(1) '9' names a ___?___ number,

and (2) '6' names a ___?___ number.

T	A	U
0	0	0
$^{-}\frac{1}{2}$	$\frac{1}{2}$	$^{+}\frac{1}{2}$
$^{-}1$	1	$^{+}1$
$^{-}1\frac{1}{2}$	$1\frac{1}{2}$	$^{+}1\frac{1}{2}$
$^{-}2$	2	$^{+}2$
$^{-}3$	3	$^{+}3$
$^{-}3\frac{1}{2}$	$3\frac{1}{2}$	$^{+}3\frac{1}{2}$
$^{-}4$	4	$^{+}4$
$^{-}5$	5	$^{+}5$
$^{-}6$	6	$^{+}6$
$^{-}6\frac{1}{2}$	$6\frac{1}{2}$	$^{+}6\frac{1}{2}$
$^{-}7$	7	$^{+}7$
$^{-}8$	8	$^{+}8$
$^{-}9$	9	$^{+}9$
$^{-}10$	10	$^{+}10$
$^{-}10\frac{1}{2}$	$10\frac{1}{2}$	$^{+}10\frac{1}{2}$
$^{-}11$	11	$^{+}11$
$^{-}12$	12	$^{+}12$
⋮	⋮	⋮

Table 14

[Part 105] [Page 102]

If we wanted to state the fact that

(A) $^{+}9 + {}^{-}3 = {}^{+}6$

and wrote:

(B) $9 + {}^{-}3 = 6$

then, we should agree that

(1) '9' names a *nonnegative (real)* number, and

(2) '6' names a *nonnegative (real)* number.

Notice that sentence (B), above, uses NAMES of numbers of arithmetic in referring to nonnegative numbers. The fact that there is a '$^{-}$' in the sentence tells us that the sentence is about real numbers.

Record your results.

* * *

Anyone who looks at:

$9 + {}^{-}3 = 6$

and believes that it should make sense

must conclude that

'9' and '6'

are numerals for positive numbers, rather than numerals for numbers of arithmetic.

He would conclude this because

it would not make sense

(1) to add a ___?___ number

(2) to a number of ___?___.

T	A	U
0	0	0
$^{-}\frac{1}{2}$	$\frac{1}{2}$	$^{+}\frac{1}{2}$
$^{-}1$	1	$^{+}1$
$^{-}1\frac{1}{2}$	$1\frac{1}{2}$	$^{+}1\frac{1}{2}$
$^{-}2$	2	$^{+}2$
$^{-}3$	3	$^{+}3$
$^{-}3\frac{1}{2}$	$3\frac{1}{2}$	$^{+}3\frac{1}{2}$
$^{-}4$	4	$^{+}4$
$^{-}5$	5	$^{+}5$
$^{-}6$	6	$^{+}6$
$^{-}6\frac{1}{2}$	$6\frac{1}{2}$	$^{+}6\frac{1}{2}$
$^{-}7$	7	$^{+}7$
$^{-}8$	8	$^{+}8$
$^{-}9$	9	$^{+}9$
$^{-}10$	10	$^{+}10$
$^{-}10\frac{1}{2}$	$10\frac{1}{2}$	$^{+}10\frac{1}{2}$
$^{-}11$	11	$^{+}11$
$^{-}12$	12	$^{+}12$
⋮	⋮	⋮

Table 14

[Part 105] [Page 103]

Anyone who looks at '$9 + {}^{-}3 = 6$', and believes that it should make sense,

must conclude that

'9' and '6' are numerals for positive numbers.

He would conclude this because it would not make sense

(1) o add a *positive (real)* number

(2) to a number of *arithmetic*.

Record your results.

* * *

So, when you look at:

$$9 + {}^{-}2 = 7,$$

you think of it as:

$${}^{+}9 + {}^{-}2 = {}^{+}7$$

And, writing:

$$9 + {}^{-}2 = 7$$

instead of:

$${}^{+}9 + {}^{-}2 = {}^{+}7$$

saves you the trouble of writing the little plus signs.

Consider another example:

$$7 \times 4 = 28$$

(1) Is the sentence above about numbers of arithmetic? (yes/no)

(2) Is the sentence above about real numbers? (yes/no)

T	A	U
0	0	0
${}^{-}\frac{1}{2}$	$\frac{1}{2}$	${}^{+}\frac{1}{2}$
${}^{-}1$	1	${}^{+}1$
${}^{-}1\frac{1}{2}$	$1\frac{1}{2}$	${}^{+}1\frac{1}{2}$
${}^{-}2$	2	${}^{+}2$
${}^{-}3$	3	${}^{+}3$
${}^{-}3\frac{1}{2}$	$3\frac{1}{2}$	${}^{+}3\frac{1}{2}$
${}^{-}4$	4	${}^{+}4$
${}^{-}5$	5	${}^{+}5$
${}^{-}6$	6	${}^{+}6$
${}^{-}6\frac{1}{2}$	$6\frac{1}{2}$	${}^{+}6\frac{1}{2}$
${}^{-}7$	7	${}^{+}7$
${}^{-}8$	8	${}^{+}8$
${}^{-}9$	9	${}^{+}9$
${}^{-}10$	10	${}^{+}10$
${}^{-}10\frac{1}{2}$	$10\frac{1}{2}$	${}^{+}10\frac{1}{2}$
${}^{-}11$	11	${}^{+}11$
${}^{-}12$	12	${}^{+}12$
⋮	⋮	⋮

Table 14

[Part 105] [Page 104]

YES, either:

$$7 \times 4 = 28$$

is a statement about numbers of arithmetic

OR

it is a statement about real numbers,

but, you can't tell which and

neither can anyone else!

You may take your choice, but not more than one of the two questions should be answered with a 'yes'.

Record your results.

* * *

Unless we know the problem which led to someone's writing:

$$7 \times 4 = 28$$

we are free to interpret it either way.

Numerals such as those in:

$$7 \times 4 = 28$$

which name more than one number [number of arithmetic or real number] are said to be ambiguous.

Pronounce 'ambiguous' as you would pronounce 'am-BIG-you-us'.

We have already used one ambiguous numeral:

'0' names { the number 0 of arithmetic or the real number 0.

Since '0' names more than one number, we say '0' is an ___?___ numeral.

T	A	U
0	0	0
${}^{-}\frac{1}{2}$	$\frac{1}{2}$	${}^{+}\frac{1}{2}$
${}^{-}1$	1	${}^{+}1$
${}^{-}1\frac{1}{2}$	$1\frac{1}{2}$	${}^{+}1\frac{1}{2}$
${}^{-}2$	2	${}^{+}2$
${}^{-}3$	3	${}^{+}3$
${}^{-}3\frac{1}{2}$	$3\frac{1}{2}$	${}^{+}3\frac{1}{2}$
${}^{-}4$	4	${}^{+}4$
${}^{-}5$	5	${}^{+}5$
${}^{-}6$	6	${}^{+}6$
${}^{-}6\frac{1}{2}$	$6\frac{1}{2}$	${}^{+}6\frac{1}{2}$
${}^{-}7$	7	${}^{+}7$
${}^{-}8$	8	${}^{+}8$
${}^{-}9$	9	${}^{+}9$
${}^{-}10$	10	${}^{+}10$
${}^{-}10\frac{1}{2}$	$10\frac{1}{2}$	${}^{+}10\frac{1}{2}$
${}^{-}11$	11	${}^{+}11$
${}^{-}12$	12	${}^{+}12$
⋮	⋮	⋮

Table 14

Note the use of our pronunciation device in lines 8 and 9 from the bottom on page 104.

PART 106

Part 106 is devoted to developing conventions for interpreting expressions from which certain of the grouping symbols have been omitted. This is tantamount to agreeing on an order of operations for an expression.

A "story problem" is employed to make the students aware of the need for reaching some agreements if we are to know whether '2 + 8 × 5' names 42 or 50.

[Part 106] [Page 1]

Bill was asked to find out how many seats there were in Room 207. He reported to the class that there were 42 seats in Room 207. He wrote the following on the board:

$$2 + 8 \times 5$$

and said that the class could check his computation.

Steve objected. He said that when he worked it out, he got 50. He justified his answer in the following manner:

"First, I found the sum of __?__ and __?__.

Then I multiplied that sum by __?__."

Fill the blanks on your work sheet to complete the sentences.

[Part 106] [Page 2]

Check your answers.

Bill claimed that there were 42 seats in Room 207.

$2 + 8 \times 5$ is 42.

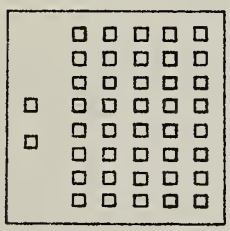
Steve said that $2 + 8 \times 5$ is 50.

Steve first found the sum of __2__ and __8__, and then he multiplied this sum by __5__.

Record your results.

* * *

Here is the floor plan for Room 207, the room Bill went to.



Who was right, Bill or Steve?
Circle the answer on your work sheet.

The problems on page 4 point out to the student that grouping symbols are indeed a very important part of an expression. The problems on page 6 point out again that, without grouping symbols or some agreement, certain numerical expressions could be used to name more than one number [a very undesirable feature].

[Part 106] [Page 4]

Check your answer.

Bill wrote: $2 + 8 \times 5$

Bill meant: $2 + (8 \times 5) = 2 + 40 = 42$

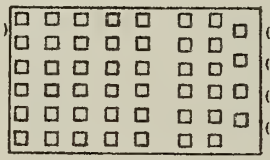
Steve thought: $(2 + 8) \times 5 = 10 \times 5 = 50$

Record your result.

* * *

Here are seating plans for two class rooms. For each of them, pick the expression which names the number of seats in the room. Circle the correct choices on your work sheet.

(1)



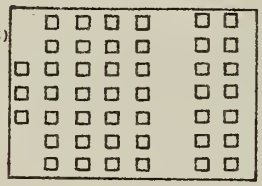
(a) $(5 + 2) \times (6 + 4)$

(b) $[(5 \times 6) + (2 \times 6)] + 4$

(c) $[5 + (2 \times 6)] + 4$

(d) $[(6 \times 5) + 2] + 4$

(2)



(a) $3 + [(7 \times 4) + 2]$

(b) $3 + [(7 \times 2) + 4]$

(c) $3 + [7 \times (4 + 2)]$

(d) $[3 + (4 + 2)] \times 7$

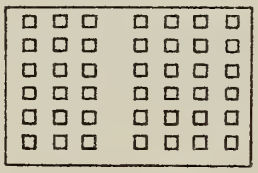
[Part 106] [Page 6]

Here are the seating plans for two other rooms in the school. Below each is an expression a student used to compute the number of seats in the room.

(A)

$6 \times 3 + 4$

(B)



$6 \times 3 + 4$

Study the charts and related expressions. It may be clear just what number '$6 \times 3 + 4$' names when looking at seating plan (A) or at seating plan (B). However, '$6 \times 3 + 4$' is <u>ambiguous</u> because it could name more than one number.

Complete the following sentences by filling the blanks on your work sheet.

(1) When related to plan (A), $6 \times 3 + 4 =$ __?__. You compute this by first finding the product of 6 by __?__, and then adding __?__ to this product.

(2) When related to plan (B), $6 \times 3 + 4 =$ __?__. You compute this by first finding the sum of 3 and __?__, and then multiplying __?__ by this sum.

On page 7 the students meet a grouping problem in an English sentence. Such a problem gives the student a "feeling" for the problem faced in mathematical sentences.

On page 12 the students are introduced to the Simple-Minded Solution Generator, a machine which performs binary operations. The problems on pages 13, 14, and 15 are designed to make the students aware of how the machine works.

[Part 106] [Page 7]

Check your answers.

(1) Related to plan (A), $6 \times 3 + 4 =$ 22. You compute this by first finding the product of 6 by 3, and then adding 4 to this product.

(2) When related to plan (B), $6 \times 3 + 4 =$ 42. You compute this by first finding the sum of 3 and 4, and then multiplying 6 by this sum.

Record your results.

* * *

Here is a case of an English sentence which is ambiguous:

John will play and Bill will sing or Mary will sing.

This sentence could mean that:

(1) Either John will play and Bill will sing or Mary will sing,

or it could mean that:

(2) John will play and either Bill will sing or Mary will sing.

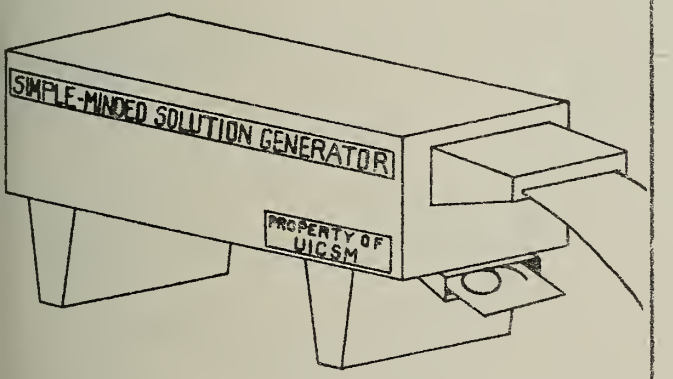
To make an English sentence unambiguous you usually have to rewrite it and use grouping words like 'either --- or ---'.

Use the grouping words 'either --- or ---' and rewrite the following sentence in two ways to make it unambiguous in each case:

Bill will be elected Student Council president and John will be elected vice-president or Mary will be elected secretary.

Write each of the new sentences on your work sheet.

[Part 106] [Page 12]



Much of our recent scientific progress is due to the use of high speed electronic computers for making computations. Let's pretend that the computer pictured above can add, subtract, multiply, and divide, but before it will perform any of these BINARY operations, it must be told to do so. One "tells" his problem to this computer by typing the expression to be computed on paper tape which is then fed into the computer.

[Part 106] [Page 13]

Suppose that we wish the computer to add 61 and 58--that is, to perform the binary operation addition on the pair (61, 58). We prepare a tape like this:

61 + 58

and feed it into the computer.

It is helpful to think of this computer as having an "eye", a device which "reads" the information on the tape as the tape passes through the machine. Pictured here is a diagram of the "eye":

Computer's "Eye" [ ] 61 + 58

The eye is limited in the sense that it can only read one character, or bit of information, at a time. The first thing the computer reads on the above tape is the '6'. This computer just doesn't know what to do with the '6'; so, it shuts itself off.

Our computer must have some signal to tell it that a binary operation is coming up. The signal used with this computer is the left parenthesis [or bracket, or brace] written before the '6'. So the tape might look like this:

[ ] (61 + 58 or: [61+ or: {61+

Now, as the tape moves past the "eye", the first thing it reads is:

[(] 61 + 58

The computer now knows that a ___?___ operation is coming up. Fill in the blank on your work sheet.

[Part 106] [Page 14]

Check your answer and record your result.

When the computer reads the left parenthesis:

(61 + 58

it knows that a binary operation is coming up.

* * *

Still concerning the tape:

(61 + 58

here is how the computer interprets the first five characters written on the tape:

(1) [(] ⟸ *Binary operation coming up.*

(2) [6] ⟸ *SIX. Perhaps the first number is 6, perhaps it's 649. Read some more.*

(3) [1] ⟸ *ONE. Better keep on reading.*

(4) [+] ⟸ *Operator '+'. So the binary operation is addition and the first addend is 61.*

(5) [5] ⟸ *FIVE. Maybe the second addend is 5, maybe it's 52. Keep reading.*

[8] When the computer reads the '8', does it then know enough to perform the binary operation?

Circle the correct answer on your work sheet.

[Part 106] [Page 15]

Check your answer.

No! The computer can't be sure that it has all of the information. As far as it can tell, the second addend might be 583, 58.72, or any other number whose name begins with '58'.

Record your result.

* * *

As we have seen, the computer needs a signal, like the left parenthesis, to tell it that it is to perform a binary operation. The computer must also be told that it has read all the information it needs to perform the binary operation signaled by the left parenthesis.

So, if we wish the computer to add 61 and 58, we would prepare a tape like this:

(61 + 58)

and feed it into the computer. The right parenthesis tells it to compute.

The answer the computer gets when it adds 61 and 58 is: ___?___

Fill in the blank on your work sheet.

The problems on page 27 lead the student through an analysis of each symbol in a given expression as they follow the Simple-Minded Solution Generator's "eye" as it reads:

[6 + (18 × 5)]

[Part 106] [Page 27]

Fill the blanks on your work sheet.

Follow the computer's "eye" as it reads: [6 + (18 × 5)].

(1) [[] ___?___ ___?___ *coming up.*

(2) [6] *SIX. Keep reading.*

(3) [+] *Binary operation is* ___?___, *and the first addend is* ___?___.

(4) [(] *A second* ___?___ *operation coming up. Complete this before doing the operation associated with the brackets.*

(5) [1] *ONE. Keep reading.*

(6) [8] *EIGHT. Keep reading.*

(7) [×] *The operation associated with the left parenthesis is* ___?___. ___?___ *is the first factor.*

(8) [5] *FIVE. Keep reading.*

(9) [)] *Now perform the operation* ___?___. *Find the product of 18 by* ___?___.

(10) []] *Perform the operation* ___?___. *Problem completed.*

On page 34 the student should become aware of the relation of grouping symbols to order of operations.

The idea of the Principal Operator is introduced on pages 37 through 39. It will be used to introduce the first convention for omitting grouping symbols. [Page 47]

[Part 106] [Page 34]

Consider the following expression for the computer to read and simplify.

[(7 × 3) + (5 × 3)]

grouping symbols associated with '?'

[(7 × 3) + (5 × 3)]

grouping ___?___ associated with '?'

The computer simplifies '[(7 × 3) + (5 × 3)]' as follows:

[(7 × 3) + (5 × 3)]

[___?___ + (5 × 3)]

[___?___ + ___?___]

Result ___?___

The last operation to be performed in order to simplify this expression is ___?___.

Fill the blanks on your work sheet.

[Part 106] [Page 37]

The last operation to be performed in order to simplify an expression is called the principal operation. The binary operator [+, −, ×, or ÷] which indicates the principal operation is called the PRINCIPAL OPERATOR.

Note that the principal operator is the operator associated with the OUTERMOST grouping symbols.

For example, the last operation performed in simplifying '[(7 + 3) × (5 × 6)]', was the operation associated with the ___?___ [brackets/parentheses]. Hence, the principal operation is ___?___ and the principal operator is '___?___'.

Fill in the blanks.

[Part 106] [Page 38]

Check your answers.

The last operation performed to simplify '[(7 + 3) × (5 × 6)]' was the operation associated with the *brackets*. Hence, the principal operation is *multiplication* and the principal operator is '×'.

Record your results.

* * *

Now that you have some knowledge of how the computer "reads" expressions and computes with them, perhaps you can put this knowledge to some use. Here are some expressions. Simplify them and name the principal operator.

	Expression	(a) Result	(b) Principal operator
(1)	[(3 + 41) × 6]	___?___	'___?___'
(2)	{(16 × 5) + (7 × 5)]	___?___	'___?___'
(3)	{[(8 + 12) × 3] + 5}	___?___	'___?___'

Fill the blanks on your work sheet.

[Part 106] [Page 39]

Check your answers.

	(a) Result	(b) Principal operator
(1) [(3 + 41) × 6]	= 44 × 6	
	= 264	'×'
(2) [(16 × 5) + (7 × 5)]	= [80 + 35]	
	= 115	'+'
(3) {[(8 + 12) × 3] + 5}	= {[20 × 3] + 5}	
	= {60 + 5}	
	= 65	'+'

Record your results.

* * *

Here are some more expressions for you. Simplify them and name the principal operator.

	(a) Result	(b) Principal operator
(1) [(5 + ⁻1) × (8 + ⁻2)]	?	'?'
(2) {[(5 + ⁻1) × 8] + ⁻2}	?	'?'
(3) [(9 × 3) + (5 + 13)]	?	'?'
(4) [(3 + 5) × (4 + 7)]	?	'?'
(5) [(16 + 7) × 5]	?	'?'

Fill in the blanks on your work sheet.

Pages 45 and 46 afford the students an opportunity to evaluate their own progress to this point.

[Part 106] [Page 45]

SELF-QUIZ 1

Write the answers to these questions on your work sheet.

1. Punctuate each of the following expressions so that our computer will get the stated result.

Expression	Result
(a) 8 × 5	40
(b) 3 × 7 + 2	23
(c) 4 × 5 + 2	28
(d) 2 ÷ 6 ÷ 3	1
(e) 2 × 3 + 4 × 5	26
(f) 2 × 3 + 4 × 5	70

2. For each of the following expressions, draw an arrow which points to the principal operator. The expressions are repeated on your work sheet.

(a) [(5 × 2) + 3] (b) [3 + (2 + 8)]

(c) [5 × (2 + 3)] (d) [(3 + 2) + 8]

(e) [(× 5) ÷ (+ 8)]

[Part 106] [Page 46]

3. Here are some expressions which are not completely punctuated. But, for each expression, the arrow points to the principal operator. With this information you can simplify it. Do so.

Expression	Simplified result
(a) 3 + 4 × 2	?
(b) 2 × 5 + 8 ÷ 4	?
(c) 2 + 5 + 9	?
(d) 2 + 5 + 9	?
(e) (104 + 16) + (92 × 37) × (85 - 85)	?
(f) (104 + 16) + (92 × 37) × (85 - 85)	?

Now that you have finished this quiz, turn to PAGE 76 and check your answers.

In the remainder of Part 106, the students are required to write many expressions using a set of grouping symbols for each operator. This creates a desire on their part to adopt some conventions for omitting grouping symbols. The conventions adopted are stated on pages 47, 59, and 70. After each convention is adopted, the students are given practice in applying it.

[Part 106] [Page 47]

You have been working with expressions like:

$[(24 \div 3) \times (6 \div 3)]$ and: $[(48 - 16) \div (16 - 5)]$

As you can see, these expressions are quite complicated. There ought to be some way in which we can leave out some of the grouping symbols and still keep the expressions from being ambiguous.

Consider the outermost grouping symbols. The main function of these symbols was to tell the computer when to start, and when to stop reading an expression. Remember that the computer's "eye" sees only one bit of information at a time. The human eye, on the other hand, can see many of these bits at once. In fact, the space on either side of an expression should be enough to set it apart. So, the first agreement, or CONVENTION, that we will make is this:

> Outermost grouping symbols may be omitted from expressions.

[Part 106] [Page 59]

Check your answers.

We punctuate '30 - 9 - 7' to name 14 as: (30 - 9) - 7

We punctuate '30 - 9 - 7' to name 28 as: 30 - (9 - 7)

Record your results.

* * *

Up to this point, we have told you about one convention for omitting grouping symbols. We shall now tell you about other conventions.

Consider the expression: 30 - 9 - 7. As we have seen, this expression can be punctuated to name 14 and to name 28. To simplify this expression, two subtractions must be performed. The two binary operators '-' indicate this. Because we read from left to right, let's AGREE to perform these operations in the same order, from left to right. So, the second convention, which we call our LEFT-TO-RIGHT convention, is this:

> To simplify expressions which do not contain grouping symbols and which have the binary operator '-' as the only operator, we will agree that the operations be performed from left to right.

Therefore, when we write this: 30 - 9 - 7
we mean ⟶ (30 - 9) - 7

In a similar manner: 40 - 23 - 15
is an abbreviation for: ?

Fill the blank on your work sheet.

[Part 106] [Page 70]

Check your answers.

(1) $30 \div 2 \div 5 = 15 \div 5 =$ 3

(2) $39 - 15 - 8 - 10 = 24 - 8 - 10 =$ 6

(3) $217 - 117 - 32 = 100 - 32 =$ 68

(4) $260 \div 10 \div 13 \div 2 = 26 \div 13 \div 2 = 2 \div 2 =$ 1

(5) $89 - 15 - 31 - 12 - 5 = 74 - 31 - 12 - 5 = 43 - 12 - 5 =$ 26

Record your results.

* * *

Recall that Bill solved his problem of punctuating expressions like '64 ÷ 8 ÷ 4' by using ideas similar to those in the left-to-right convention. We shall make a convention similar to this which will include each of the binary operators: +, -, ×, and ÷. So, let's just re-word our present left-to-right convention so that it will include these four binary operators. Hence, the new LEFT-TO-RIGHT convention will now be:

> Given expressions which do not contain grouping symbols, and which contain only operators of one kind, we agree to simplify these expressions by working from left to right.

For example, consider: 25 + 35 + 9 + 17
Using the LEFT-TO-RIGHT convention to simplify this, we have:

25 + 35 + 9 + 17
= ? + 9 + 17
= ? + 17
= ? ⟵ Result

Fill the blanks on your work sheet.

Page 75 affords the students a second opportunity to evaluate their progress in this part.

SELF-QUIZ II.

1. Punctuate according to convention and simplify. [Leave out the outermost grouping symbols.]

   (a) $5 \times 3 \times 4$

   (b) $240 \div 6 \div 8$

   (c) $168 - 58 - 30 - 8$

   (d) $38 + 72 + 51 + 10$

   (e) (3×2)

2. Fill in the blanks to get the result shown.

   (a) $(16 \div 2) + (21 \div \underline{\ ?\ }) = 11$

   (b) $(5 \times \underline{\ ?\ } \times 2) - (18 \div 2 \div 3) = 37$

   (c) $18 + 7 + \underline{\ ?\ } + 5 = 47$

   (d) $(94 \times \underline{\ ?\ }) + (6 \times 38) = 100 \times 38$

   (e) $(5 \times \underline{\ ?\ }) - (\underline{\ ?\ } \times 5) = 0$

3. For each of the following expressions, draw an arrow which points to the principal operator.

   (a) $3 \times 9 \times 12$ (b) $3 + 9 + 12$

   (c) $10 - 6 - 2$ (d) $1 \div 5 \div 6$

   (e) $(3 + 5) \times (8 + 6) \times (7 + 9)$

   (f) $(2 \times 7) + [(8 \times 6) + (3 \times 9)]$

Now that you have finished this quiz, turn to PAGE 78 and check your answers.

PART 107

At this point the students have reached agreements for omitting grouping symbols when an expression contains only one kind of operator. Although it is the tradition of UICSM not to state "rules" for the students, we feel justified in stating the conventions for grouping symbols since these are merely writing conventions and not mathematical principles.

In Part 107 the students are introduced to the following conventions.

> In expressions which contain the two kinds of operators, '+' and '-', we will simplify by performing the operations, as indicated, from LEFT to RIGHT.

> In expressions which contain the two binary operators, '×' and '÷', we shall simplify by performing the operations, as indicated, from LEFT to RIGHT.

> In simplifying unpunctuated expressions which contain a multiplication sign and an addition sign, we shall PERFORM THE MULTIPLICATION FIRST.

> In simplifying unpunctuated expressions which contain the binary operators '+' and '×', we agree to PERFORM THE MULTIPLICATIONS FIRST from left to right, in order, and then PERFORM THE ADDITIONS from left to right.

> In simplifying unpunctuated expressions which contain the operators '×', '+', and '-', perform the MULTIPLICATIONS FIRST, from left to right, and then perform the ADDITIONS AND SUBTRACTIONS from left to right.

In simplifying unpunctuated expressions which contain several indicated divisions and several indicated additions, we shall FIRST PERFORM THE DIVISIONS from left to right, in order, and then PERFORM THE ADDITIONS from left to right.

In simplifying unpunctuated expressions which contain the operators '÷', '+', and '-', perform the DIVISIONS FIRST, from left to right, and then perform the ADDITIONS AND SUBTRACTIONS from left to right.

To simplify unpunctuated expressions which contain the operators '×', '÷', and '+', first perform the multiplications and divisions from left to right, and then perform the additions in the same order.

To simplify unpunctuated expressions which contain indicated multiplications or divisions and indicated additions or subtractions, we will agree to perform first the multiplications and divisions from left to right, and then perform the additions and subtractions from left to right.

Given expressions which contain operators, some of which are paired with grouping symbols, and some of which are not, we will simplify these expressions by first performing the operations paired with the grouping symbols, The resulting expressions can then be simplified by use of our prior conventions.

On the intervening pages, the students are given ample opportunity to apply these conventions and are now equipped to interpret any type of numerical expression.

Pages 70-72 consider the meaning of:

fraction

We feel that the following problem, taken from the article " "1" and "1" is "11" " by Frank L. Wolf in the April 1958 issue of The Arithmetic Teacher, should point out why we wish to regard a fraction as a numeral rather than as a number.

Mr. Wolf asks the reader to tell what is wrong with the following reasoning:

Since 5 is a divisor of 10, we know that 5 is a divisor of the numerator of $\frac{10}{15}$. Hence, because $\frac{10}{15} = \frac{2}{3}$ and because we may substitute equals for equals, 5 is a divisor of 2.

!

[Part 107] [Page 70]

Check your answers.

	Expression (a)	Result (a)	Expression (b)	Result (b)
(1)	5 + 2	7	5 × 2	10
(2)	3 + 4	7	$2\frac{1}{2}$ × 4	10
(3)	4 + 3	7	$3\frac{1}{3}$ × 3	10
(4)	2 + 5	7	2 × 5	10

Record your results.

* * *

Consider the expression: $\frac{1}{4}$

An expression like the given one is called a fraction. A fraction is a numeral which consists of a fraction bar [—], a numeral written above the fraction bar [in this case: 1], and a numeral written below the fraction bar [in this case: 4].

The expression written above the fraction bar is called the numerator. The expression written below the fraction bar is called the denominator.

In the given expression: $\frac{1}{4}$

the numerator is: ?

and the denominator is: ?

Fill the blanks on your work sheet.

[Part 107] [Page 71]

Check your answers.

In the expression: $\frac{1}{4}$

the numerator is: 1 (a)

the denominator is: 4 (b)

Record your results.

* * *

Tell the numerator and denominator of each of the following fractions by filling the blanks on your work sheet.

	(a) Numerator:	(b) Denominator:
(1) $\frac{4}{(12 - 3)}$	?	(12 - 3) [not: 9]
(2) $\frac{8 \times 3}{12 + 8}$	?	?
(3) $\frac{(6 \times 3) + (2 \times 3)}{12 + 8}$	?	?
(4) $\frac{16 \div 4}{3 \times 4}$	?	?

[Part 107] [Page 72]

Check your answers.

	(a) Numerator	(b) Denominator
(1) $\frac{4}{(12 - 3)}$	4	(12 - 3)
(2) $\frac{8 \times 3}{12 + 8}$	8 x 3	12 + 8
(3) $\frac{(6 \times 3) + (2 \times 3)}{12 + 8}$	(6 x 3) + (2 x 3)	12 + 8
(4) $\frac{16 \div 4}{3 \times 4}$	16 ÷ 4	3 x 4

The numerator is precisely the expression written above the fraction bar. For example, you are incorrect if you say the numerator in problem 2 is '24'. Record your results.

* * *

Consider the expression: $\frac{8 \times 3}{12 + 8}$

We simplify this expression by simplifying the numerator and the denominator. So:

$$\frac{8 \times 3}{12 + 8} = \frac{24}{20} = \frac{6}{5}$$

Simplify each of the following expressions by filling the blanks on your work sheet.

(1) $\frac{16 \div 4}{3 \times 4}$ = ?

(2) $\frac{4}{12 - 3}$ = ?

(3) $\frac{(6 \times 3) + (2 \times 3)}{12 + 8}$ = ?

Pages 73 and 74 point out that the fraction bar can be considered as a grouping symbol in an expression such as:

$$\frac{8 - 7}{12 - 8}.$$

where the principal operation is division.

[Part 107] [Page 73]

Check your answers.

(1) $\frac{16 \div 4}{3 \times 4} = \frac{1}{3}$

(2) $\frac{4}{12 - 3} = \frac{4}{9}$

(3) $\frac{(6 \times 3) + (2 \times 3)}{12 + 8} = \frac{18 + 6}{20} = \frac{6}{5}$

Record your results.

* * *

Up to this time we have discussed the use of parentheses, brackets, and braces as grouping symbols in expressions. One might also think of the fraction bar as a grouping symbol in an indicated division problem. That is, one might think of the fraction as an abbreviation for an expression in which the principal operator is '÷'.

For example, the fraction '$\frac{1}{4}$'
might be thought of as an abbreviation for
'1 ÷ 4'

Also, the fraction '$\frac{8 - 7}{12 - 8}$'
might be thought of as an abbreviation for
'(8 - 7) ÷ (12 - 8)'

In a similar manner, the expression
'$\frac{8 \times 3}{12 + 8}$'
might be an abbreviation for
'(?) ÷ (?)'

Fill the blank on your work sheet.

[Part 107] [Page 74]

Check your answer.

The expression '$\frac{8 \times 3}{12 + 8}$'
might be thought of as an abbreviation for
'(8 x 3) ÷ (12 + 8)'

Record your result.

* * *

Let's agree to consider that each fraction is an abbreviation for an expression in which the principal operator is '÷'.

Each of the following expressions is an abbreviation for an expression for which the principal operator is '÷'. Write the unabbreviated expressions in the blanks on your work sheet.

(1) $\frac{6 + 2}{8}$ = ?

(2) $\frac{(6 + 2) \times 5}{8}$ = ?

(3) $\frac{6 \times 2 + 5}{8 + 9}$ = ?

(4) $\frac{5 + 6 \div 2}{5 + 3}$ = ?

This next section consists of excerpts from Parts 108, 109. 110, and 110.5. Here are some sample test items from these parts.

You have five minutes to simplify as many of the following numerical expressions as you can.

$$21 \times 41 \times 25 \times \frac{1}{3} \times \frac{1}{5}$$

$$^{-}489 \times {}^{-}36 + {}^{-}489 \times {}^{+}36$$

. . .

Indicate in the blanks for each of the following whether it is an <u>instance</u> of a principle or a <u>consequence</u> of a principle and tell which principle.

$$(4 \times 3) + 6 + 8 = (4 \times 3) + (6 + 8)$$

$$6 \times 5 + 2 + 3 = 2 + 6 \times 5 + 3$$

. . .

Fill in the blanks to make a proof of the given statement.

$$\underline{1 \times (1 + 0) = 1}$$

$1 \times (1 + 0) =$ _____	(because of the)	[_____]
$1 \times 1 + 1 \times 0 =$ _____		[pmo]
_____ $= 1 + 0$		[_____]
$1 + 0 = 1$		[pao]

. . .

PART 108

It is generally accepted that a modern treatment of elementary algebra must include a deductive system based on the field postulates for the real numbers. Without it, there is no mathematical rationale for manipulative techniques.

On the following pages you will see the unique UICSM approach to these postulates. We develop awareness of them in the student by helping the student discover shortcuts for simplifying expressions involving numbers of arithmetic. (Later, he makes the transition to real numbers.) Notice that letter variables are avoided, since they do not contribute to the student's belief in the postulates. In fact, no complete statement of the postulates appears until Unit II (not included in this programing project). We shall refer to the generalizations discovered through patterns as principles.

The first principle is the commutative principle for multiplication.

[Part 108] [Page 1]

The refreshments committee of the Zabranchburg High School Mathematics Club bought eight 12-ounce bottles of cola. How many 8-ounce glasses of cola can they serve?

[Part 108] [Page 2]

Check your answer.

Here is one way to solve the problem:

Suppose that we pour one 8-ounce glass from each one of the eight 12-ounce bottles of cola.

Each bottle now has 4 ounces left in it; so, we can fill one 8-ounce glass from each two bottles.

Now our 8 twelve-ounce bottles are empty, and we have filled 12 eight-ounce glasses.

Of course, if you got the same answer in some other way, consider yourself correct. Record your result.

* * *

Mr. Cisum, the bandleader at Zabrachburg High School, had the band march out to the football field 3 abreast in 17 rows. At the edge of the field he blew his whistle 3 times as a signal to the band to form 3 long rows. Then, they marched across the field. How many band members were in each row?

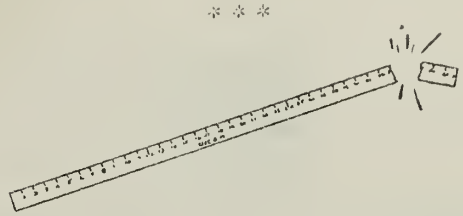
[Part 108] [Page 10]

Check your answer.

Since Mr. Wisehart planted 20 of the 80 acres in clover, he had only 60 acres left on which to plant corn. Last year he produced 4800 bushels of corn [80 acres at 60 bushels per acre]. So, he must produce 4800 bushels of corn this year [60 acres at 80 bushels per acre].

Record your result.

* * *



Sammy accidentally broke a 3-inch piece off his yardstick and threw the shorter piece away.

(1) How long is the part of the yardstick that Sammy kept?

(2) If Sammy measures a length of 33 yards with his broken yardstick by repeatedly laying it end to end, how many times must he lay it down?

[Part 108] [Page 11]

Check your answers.

(1) His broken yardstick is 33 inches long.

(2) The length which Sammy wishes to measure is 1188 inches [33 yards at 36 inches per yard]. To find the number of times he must lay down his "33-inch stick" end to end, we could divide 1188 by 33.

```
      36
33)1188
   99
    198
    198
```

Record your results.

* * *

Here is part of a division chart:

÷	2	3
12	6	4
6		2

Answer these questions on your work sheet.

(1) This portion of the chart tells us that 12 ÷ 2 = ?.

÷	2	
12	6	

(2) This portion of the chart tells us that ? ÷ 3 = 2.

÷		3
6		2

(3) This portion of the chart tells us that 12 ÷ ? = 4.

÷		3
12		4

(4) Complete this chart:

÷	2	3
12	6	4
6	?	2

After several exercises of the preceding type, the student becomes acquainted with division and multiplication charts. The skill being developed through the use of these charts will be used throughout all of his mathematical training.

[Part 108] [Page 13]

Check your answers.

÷	1	2	3
1	1	$\frac{1}{2}$	$\frac{1}{3}$
2	2	1	$\frac{2}{3}$
3	3	$1\frac{1}{2}$	1

Record your results.

[Your answers may be correct even though they are not exactly the same as these. Check to see that the answer in each of your boxes is equivalent to the answer in the corresponding box here.]

* * *

÷	98	32	51
32	(a)	(b)	(c)
98	(d)	(e)	(f)
51	(g)	(h)	(i)

Suppose that the division table above was filled in correctly with numerals for the quotients. In which box [(a), (b), (c), (d), (e), (f), (g), (h), or (i)] would you find an answer to these:

(1) 98 divided by 32 = ?
(2) 32 divided by 51 = ?
(3) 51 divided by 32 = ?
(4) 98 divided by 51 = ?
(5) 32 divided by 98 = ?

[Part 108] [Page 14]

Check your answers.

÷	98	32	51
32	(a)	(b)	(c)
98	(d)	(e)	(f)
51	(g)	(h)	(i)

(1) An answer to '98 ÷ 32 = ?' would be found in box (e).
(2) An answer to '32 ÷ 51 = ?' would be found in box (c).
(3) An answer to '51 ÷ 32 = ?' would be found in box (h).
(4) An answer to '98 ÷ 51 = ?' would be found in box (f).
(5) An answer to '32 ÷ 98 = ?' would be found in box (a).

Record your results.

* * *

Here is a multiplication chart

X	47	72	39	56
43	2021	3096	1677	2408
54	2538	3888	2106	3024
31	1457	2232	1209	1736
78	3666	5616	3042	4368

Use it to answer the questions below.

Sample: 54 multiplied by 39 = ? Answer: 2016

(1) 43 multiplied by 72 = ?
(2) 78 multiplied by 39 = ?
(3) 31 multiplied by 56 = ?
(4) 54 multiplied by 47 = ?

X			39	
54			2106	

Now the student is equipped to handle an exercise like the following one. Of course, he may be alert enough that he will do no computing, or, he may yet have to do some. After the commutative principle for multiplication is an "old friend", the chart will be presented again (page 55) and the student will be asked:

How many products would someone need to compute to fill the ten empty cells in the chart?

[Part 108] [Page 15]

Check your answers.

(1) 43 × 72 = 3096
(2) 78 × 39 = 3042
(3) 31 × 56 = 1736
(4) 54 × 47 = 2538

X	47	72	39	56
43	2021	3096	1677	2408
54	2538	3888	2106	3024
31	1457	2232	1209	1736
78	3666	5616	3042	4368

Record your results.

* * *

Complete this multiplication chart on your work sheet.

X	$\frac{2}{3}$	1200	$\frac{3}{4}$	87	21
21	14		$\frac{63}{4}$	1827	441
87	58	104,400		7569	
$\frac{2}{3}$	$\frac{4}{9}$		$\frac{1}{2}$		
$\frac{3}{4}$		900	$\frac{9}{16}$	$\frac{261}{4}$	
1200	800	1,440,000			25,200

Another useful way to develop an awareness of principles is through equation solving. Here this technique is employed for the first time.

[Part 108] [Page 21]

Check your answers.

(1) $53 \times 57 = \underline{57} \times 53$

(2) $492 \times \underline{777} = 777 \times 492$

(3) $\underline{1941} \times \frac{1}{2} = \frac{1}{2} \times 1941$

(4) $12.34 \times 53\frac{7}{8} = 53\frac{7}{8} \times \underline{12.34}$

Record your results.

* * *

Here is a sentence with a blank in it:

(☆) $5 \times 8 = 4 \times$ ____

[Call this sentence '(☆)' --pronounce this as 'star'. Notice that we write its name at the left of the sentence.]

Now, if you write a numeral in the blank in (☆), you get a new sentence. For example, you could get:

(1) $5 \times 8 = 4 \times \underline{7}$

Or, another new sentence you could get from (☆) is:

(2) $5 \times 8 = 4 \times \underline{10}$

On your worksheet, tell which of sentences (1) and (2) are true, and which are false.

[Part 108] [Page 22]

Check your answers.

(☆) $5 \times 8 = 4 \times$ ____

(1) $5 \times 8 = 4 \times \underline{7}$ . . . . . false

(2) $5 \times 8 = 4 \times \underline{10}$ . . . . . true

Record your results.

* * *

When you put a numeral in the blank in a sentence such as (☆) , you get a new sentence [like (1) or (2)] which states something about numbers.

Sentence (1) states that 5×8 is 4×7. Of course, this statement is false because 5×8 is not 4×7.

Sentence (2) states that 5×8 is 4×10. This statement is true [as any Zabranchburg 3rd grader could tell you].

On your work sheet you'll find pairs of copies of a sentence. In the first copy, write a numeral in the blank so that you get a TRUE statement and in the second copy, write a numeral in the blank to get a FALSE statement.

(1) $2 \times 10 =$ _____

(2) $3 \times$ ___ $= 2 \times 9$

(3) $2 \times 2 = 4 \times$ ____

(4) __ $\times 76 = 76 \times 13$

[Part 108] [Page 23]

Check your answers.

True statements	False statements
(1) (a) $2 \times 10 = \underline{20}$	(b) $2 \times 10 = \underline{729}$
(2) (a) $3 \times \underline{6} = 2 \times 9$	(b) $3 \times \underline{19} = 2 \times 9$
(3) (a) $2 \times 2 = 4 \times \underline{1}$	(b) $2 \times 2 = 4 \times \underline{1492}$
(4) (a) $\underline{13} \times 76 = 76 \times 13$	(b) $\underline{777} \times 76 = 76 \times 13$

Of course, you probably used other numbers than we did to get false statements. In fact, for these given sentences, any number other than the one used for part (a) is correct for part (b).

Record your results.

* * *

Fill the blanks on your worksheet to make true statements.

(1) $17 \times 31 = 31 \times$ ____

(2) $9 \times$ ___ $=$ ___ $\times 9$

At this point, a rule is laid down concerning filling blanks in open sentences. Stated briefly, it is:

> All of the blanks in a sentence which have the same type of underline must be filled with numerals that are copies of each other.

This rule makes it possible for the student to generate sentences and then decide whether they are instances of a certain principle or not. This device is meant to replace at least two classroom situations:

1. Consideration of sentences proposed by students.
2. "Hand-waving" to establish the universality of the principles.

[Part 108] [Page 33]

Check your answers.

(1) 0 × 0 = 0 + 0
(2) (4+2) × 3 = 3 × (4+2)
(3) 25 × 13 = 13 × 25
(4) 0 × 1 = 1 × 0
(5) 1 × 1 = 1 × 1
(6) How many of the above statements are true? All of them.

Record your results.

* * *

Finish filling the blanks in the sentences below:

(1) 4 × 619 = ___ × ___
(2) ___ × 92 = ___ × 58
(3) ___ × ___ = 2975 × 5786
(4) 3 × ___ = (78+39) × ___
(5) Are any of the above statements false?

[Part 108] [Page 34]

Check your answers.

(1) 4 × 619 = 619 × 4
(2) 58 × 92 = 92 × 58
(3) 5786 × 2975 = 2975 × 5786
(4) 3 × (78+39) = (78+39) × 3
(5) Are any of the above statements false? no

Record your results.

* * *

On your work sheet there are two copies of each of the following sentences about numbers of arithmetic. You are to put numerals in the blanks in the first copy to make a true statement and put numerals in the second copy to make a false statement. Be sure that blanks with the same type of underline are filled with numerals that are copies of each other.

(1) ____ × ____ = 4
(2) 1 + (___ × ___) = 5
(3) 3 + ... = ...
(4) ___ × ..... = ... × ___

[Part 108] [Page 35]

Check your answers.

	True	False
(1) (a)	2 × 2 = 4	(b) 71 × 71 = 4 [Use any numb other than 2.
(2) (a)	1 + (2 × 2) = 5	(b) 1 + (739 × 739) = 5
(3)*(a)	3 + ... = ... [can't do it]	(b) 3 + 119 = 119 [Use any numb
(4) (a)	97 × 5 = 5 × 97	*(b) ___ × ..... = ... × ___ [Can't do it.]

*In Exercises (3a) and (4b) consider yourself correct if you realized you could not do it.

* * *

Each time the blanks are properly filled with numerals for numbers of arithmetic in the sentence:

___ × ..... = ... × ___

the statement generated is a ___?___ [true or false] statement.

Fill the blank on your worksheet.

The name of this principle is finally presented, and practice in writing the name and pronouncing it is provided.

This is followed by exercises in which the student writes instances of the principle. To motivate the writing of instances, an imaginary instance-writer machine called "Commumulter" is introduced. The pedagogical device makes clear to the student that writing instances is purely a mechanical job which is accomplished by following a pattern.

In a regular class this skill is usually taught with an indirect motivation based on performance-reward from the teacher. Of course, in programed material a great deal more ingenuity is required, and our "commumulter" is a bit of pedagogy that we hope accomplishes this.

[Part 108] [Page 39]

Check your answers.

Sentences formed from the pattern:

___×..... = ... ×___

	Does it fit the pattern?	Is it true or false?
(1) 81 × 94 = 94 × 81	(yes) no	(true) false
(2) 79 × 63 = 97 × 36	yes (no)	true (false)
(3) 13 × 93 = 39 × 31	yes (no)	(true) false
(4) (1 + 5) × 7 = 7 × (1 + 5)	(yes) no	(true) false
(5) (6 + 5) × 8 = (5 + 6) × 8	yes (no)	(true) false
(6) 36 × 57 = 56 × 37	yes (no)	true (false)

Record your results.

* * *

Anyone who looks at a statement about multiplication which fits the pattern-sentence:

___×..... = ...×___

and believes that it is true WITHOUT DOING ANY COMPUTING is probably aware of a useful mathematical principle called:

THE COMMUTATIVE PRINCIPLE FOR ___?___

what operation?

Write the missing word on your work sheet.

[Part 108] [Page 40]

Check your answer.

Someone who believes any statement which fits the pattern-sentence:

___×..... = ...×___

does so because he believes

THE COMMUTATIVE PRINCIPLE FOR MULTIPLICATION.

Record your results.

* * *

Learn how to pronounce the word 'commutative'. The following silly poem may help you:

We hope this poem you'll forgive,
Especially you people in Tiv.
For nothing is worse
Than having to rehearse
A slogan like: Come you t' Tiv.

Recite the poem to yourself a couple of times. The last part of the last line sounds very much like the way you ought to say the word 'commutative'.

Here is the way a dictionary tells you to pronounce this word:

com·mu'ta·tive (kŏ·mū' tā·tĭv)

Now, answer this question on your work sheet.

I believe that $73\frac{1}{2} \times 47\frac{1}{4} = 47\frac{1}{4} \times$ ___?___

because I believe the ___?___ principle

for ___?___.

[Part 108] [Page 41]

Check your answers.

I believe that $73\frac{1}{2} \times 47\frac{1}{4} = 47\frac{1}{4} \times$ $73\frac{1}{2}$

because I believe the commutative principle for multiplication.

Check your spelling of the word 'commutative', especially the double 'm'. [If you mispelled it, write the correct spelling on your work sheet.] Record your results.

* * *

There are two types of underlines shown in the sentence:

___×..... = ...×___

So, we need to choose a number pair to fill the blanks. Each pair that is chosen and used in this pattern-sentence for the commutative principle for multiplication produces a statement about those numbers that you should believe without doing any computing.

For instance:

999 × 222 = 222 × 999

Another instance:

(3 × 5) × (2 + 0) = (2 + 0) × (3 × 5)

And even this instance:

7 × 7 = 7 × 7

Can you see that each of these statements is generated from the pattern-sentence above? We call these statements

INSTANCES

of the commutative principle for multiplication.

If a statement cannot be formed by choosing some number pair and filling the blanks in the pattern sentence above, then it is not an instance of the commutative principle for multiplication.

Here are four statements. On your work sheet tell which are instances of the commutative principle for multiplication, and tell which are not.

(1) 3 × 7 = 7 × 3 (2) 92 × 34 = 34 × 92

(3) 5 × 20 = 2 × 50 (4) 42 × 42 = 42 × 42

Check your answers.

Instances of the commutative principle for multiplication.

(1) $3 \times 7 = 7 \times 3$ Yes No (2) $92 \times 34 = 34 \times 92$ Yes No

(3) $5 \times 20 = 2 \times 50$ Yes No (4) $42 \times 42 = 42 \times 42$ Yes No

not copies; copies; copies³ copies

Record your results.

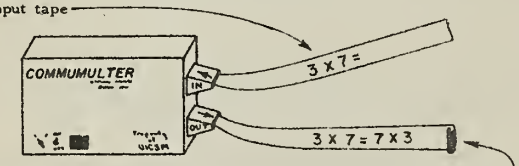
* * *

You have worked several exercises in which you were to decide whether or not a given statement is an instance of the commutative principle for multiplication. As you may have noticed, this is a rather mechanical job. All you have to do is look at the statement and notice certain things about it.

So, we have invented a machine [in our imagination] which can write instances of the commutative principle for multiplication. This is all that it can do. We call it:

COMMUMULTER

Here is how it works. Someone writes the beginning of an instance on the input tape



and Commumulter writes out the complete instance on the output tape.

Here are two input tapes for Commumulter. On your work sheet, write the corresponding output tapes.

(1) 19 x 12 = (2) 34 x 17 =

Check your answers.

	Input tape	Output tape
(1)	20 x 30 =	20 x 30 = 30 x 20
(2)	(15 + 5) x 30 =	(15 + 5) x 30 = 30 x (15 + 5)
(3)	(3 + 4) x 5 =	(3 + 4) x 5 = 5 x (3 + 4)
(4)	5 x (3 + 4) =	5 x (3 + 4) = (3 + 4) x 5

Record your results.

* * *

Here are some more input tapes. For each one, if Commumulter can't use it to write an instance of the commutative principle for multiplication, circle the 'no' on your work sheet. If it can, circle the 'yes', and write the complete instance as it would appear on the output tape.

(1) (5 + 6) x 8 =

(2) (5 + 6) x (4 + 4) =

(3) 513 x 513 =

(4) 34 + 9 x 7 =

(5) 34 x 9 + 7 =

(6) 7 x (2 x 5) =

(7) (7 x 2) x 5 =

(8) 7 x 2 x 5 =

Check your answers.

Output tape

(1) (3 x 4) x 5 = 5 x (3 x 4)

(2) 3 x (4 x 5) = (4 x 5) x 3 or: = 4 x 5 x 3

(3) 3 x 4 x 5 = 5 x (3 x 4)

(4) (3 x 4) x (5 x 6) = (5 x 6) x (3 x 4)

(5) [3 x (4 x 5)] x 6 = 6 x [3 x (4 x 5)]

(6) 3 x (4 x 5) x 6 = 6 x [3 x (4 x 5)]

(7) 6 x [3 x (4 x 5)] = [3 x (4 x 5)] x 6 or: = 3 x (4 x 5) x 6

(8) [(3 x 4) x 5] x 6 = 6 x [(3 x 4) x 5]

(9) 3 x 4 x 5 x 6 = 6 x [(3 x 4) x 5] or: = 6 x (3 x 4 x 5)

Record your results.

* * *

Someone took an output tape written by Commumulter and cut it into three pieces. He dropped the pieces on a table top like this:

On your work sheet, write the instance of the cpm that Commumulter had written on the tape.

Check your answer.

Here are the pieces of tape:

And, here is how the output tape looks when the pieces are properly put together:

8 x 3 = 3 x 8

Record your result.

* * *

Another one of Commumulter's output tapes was cut into five pieces. In dropping them on the table top, two fell face down:

On your work sheet, write an instance of the cpm which could have appeared on the output tape before it was cut. [We say 'could have appeared' because there are two possibilities. Try to get both of them.

On the next pages we review the problems which the students did earlier in the part, identifying the short cuts as applications of the commutative principle for multiplication.

Now, we shall see how the commutative principle for <u>addition</u> is introduced and developed.

[Part 108] [Page 59]

Check your answer.

$$33 \times \underline{36} = 36 \times 33$$

Record your result.

* * *

You may have noticed that in solving the last two problems, we didn't bother to simplify expressions such as '12 × 8' and '36 × 33'. It turned out that the problems were easier to solve when we used these numerals than they would have been if we had "simplified" the numerals.

* * *

Here's another Sammy-problem. Solve it.

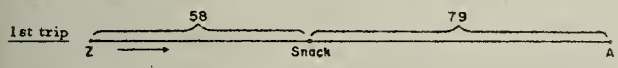
Sammy's father took a trip from Zabranchburg to Anabru. Because he had only eaten a small breakfast, he stopped for a morning snack having traveled just 58 miles. He then drove the remaining 79 miles to Anabru. The following week he took another trip to Anabru from Zabranchburg, traveling over the same route. This time, because he had eaten a larger breakfast, he traveled 79 miles before stopping for his morning snack. How many miles did he still have to drive to finish his trip to Anabru?

Answer on your work sheet.

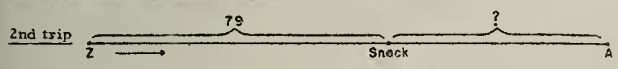
[Part 108] [Page 60]

Check your answer.

The trip from Zabranchburg to Anabru is (58 + 79) miles long.



This distance is the same no matter which week he makes the trip since he travels over the same route.



So, the problem is solved just by answering the question:

$$58 + 79 = 79 + \underline{\ ?\ }$$

Sammy's father had to travel <u>58</u> more miles to reach Anabru after his morning snack.

Record your result.

* * *

At the end of the track season last year, the Zabranchburg High team left a supply of shirts that could be used for another season.

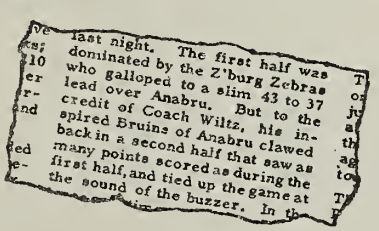
There were 28 sweatshirts and 36 T-shirts left. The coach, Mr. Young, decides to order enough new shirts in order to double the supply. He also wishes to even the supply of sweatshirts and T-shirts so there will be the same number of each kind.

Answer these questions on your work sheet.

(1) How many T-shirts should he order? (2) How many sweatshirts?

[Part 108] [Page 62]

This is part of a news item about a basketball game from the sports page of the <u>Zabranchburg Chronicle</u>.


last night. The first half was dominated by the Z'burg Zebras who galloped to a slim 43 to 37 lead over Anabru. But to the credit of Coach Wiltz, his inspired Bruins of Anabru clawed back in a second half that saw as many points scored as during the first half, and tied up the game at the sound of the buzzer. In th

How many points did Zabranchburg score in the second half, and how many did Anabru score in the second half?

Check your answers.

	Zabranchburg	Anabru
First half	37	43
Second half	43	37
	37 + 43	43 + 37

tie

Record your results.

* * *

In the last few problems you have probably noticed the fact that just as there is a commutative principle for multiplication, so there is a

(1) ____?____ PRINCIPLE FOR ____?____.

[Fill the blanks on your work sheet.]

(2) All except one of the following statements are instances of this principle. Find it.

(a) $62 + 98 = 98 + 62$

(b) $1776 + 1962 = 1962 + 1776$

(c) $73 + 81 = 83 + 71$

(d) $\frac{1}{2} + \frac{3}{4} = \frac{3}{4} + \frac{1}{2}$

Check your answers.

(1) $(4 + 3) + (5 + 9) = (5 + 9) + (4 + 3)$

(2) $(4 + 3) + (5 + 9) = (5 + 9) + (4 + 3)$

(3) $(5 + 9) + (4 + 3) = (4 + 3) + (5 + 9)$

(4) $(5 + 9) + (4 + 3) = (4 + 3) + (5 + 9)$

(5) $9 + [(4 + 3) + 5] = [(4 + 3) + 5] + 9$

(6) $9 + (4 + 3 + 5) = (4 + 3 + 5) + 9$

(7) $9 + (4 + 3 + 5) = 4 + 3 + 5 + 9$

Record your results.

* * *

On your work sheet tell whether or not each of the following statements is an instance of the commutative principle for addition, and tell if it is true or false. [It will help if you mark the principal operator and use the convention for unabbreviating.]

(1) $1 \times 0 + 0 = 0 + 1 \times 0$

(2) $1 \times 0 + 0 = 0 \times 1 + 0$

(3) $1 + 0 \times 0 = 0 + 0 \times 1$

(4) $1 + 0 \times 1 = 1 \times 0 + 1$

(5) $1 \times 0 + 0 \times 1 = 0 \times 1 + 1 \times 0$

(6) $0 \times 1 \times 0 + 1 = 1 + 0 \times 1 \times 0$

Finally, the abstract idea of commutativity of an operation is presented.

Check your answers.

(1)	$1 \times 0 + 0 = 0 + 1 \times 0$	yes no	true false
(2)	$1 \times 0 + 0 = 0 \times 1 + 0$	yes no	true false
(3)	$1 + 0 \times 0 = 0 + 0 \times 1$	yes no	true false
(4)	$1 + 0 \times 1 = 1 \times 0 + 1$	yes no	true false
(5)	$1 \times 0 + 0 \times 1 = 0 \times 1 + 1 \times 0$	yes no	true false
(6)	$0 \times 1 \times 0 + 1 = 1 + 0 \times 1 \times 0$	yes no	true false

Record your results.

* * *

As you have seen, each of the BINARY OPERATIONS addition and multiplication has a commutative principle. We say that addition and multiplication are commutative operations.

Answer the following questions on your work sheet.

(1) Is each statement generated from the pattern-sentence below a true statement?

___ ÷ ..... = ... ÷ ___

(2) True or false?

(a) $6 + 3 = 3 \div 6$

(b) $2 \div 10 = 10 \div 2$

(c) $100 + 25 = 25 \div 100$

(d) $97 \div 97 = 97 + 97$

[You may reconsider your answer to (1) if you wish!]

(3) Can each of the statements in Exercise (2) be generated from the pattern-sentence in Exercise (1)?

(4) Is division a commutative operation?

Check your answers.

(1) (a) The average of 7 and 3 is 5.
(b) The average of 3 and 7 is 5.

(2) (a) The average of 62 and 68 is 65.
(b) The average of 68 and 62 is 65.

Record your results.

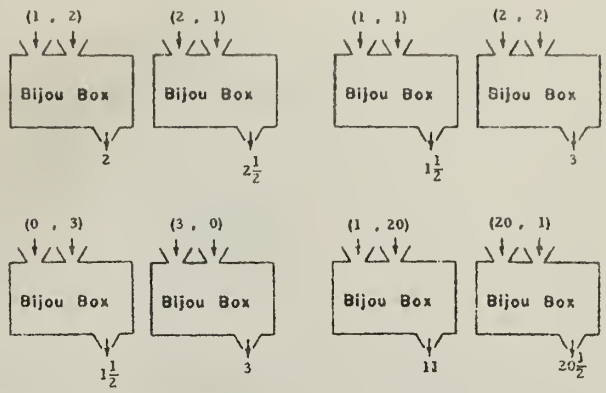
* * *

The answers to the above problems provide us with a clue as to whether or not the binary operation AVERAGING is commutative. Is it?

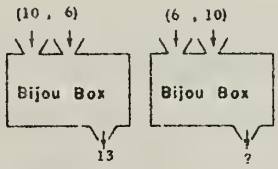
Write your answer on your work sheet.

Now turn to the other side of this page.

Study these diagrams:



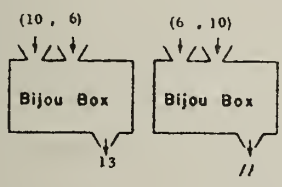
Now, you try this one:



Write your answers on the work sheet.

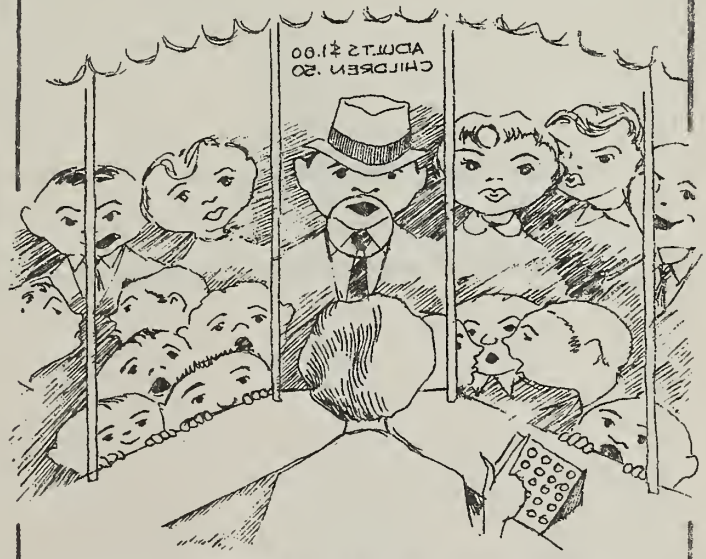
Turn to the other side of this page.

Check your answer.



Record your result.

Now read PAGE 78.



The Bijou Box is really a machine used by Sally Simpson, cashier at the Bijou Theater in Zabranchburg. At the Bijou, tickets for adults are one dollar each and tickets for children are a half-dollar each. When one person buys tickets for a large party of, say, 6 adults and 10 children, Sally merely presses the '6'-button first and then the '10'-button. The '11' that appears in a little window on the machine shows that the price for this party is 11 dollars.

Is the binary operation which the Bijou Box "performs" on pairs of numbers commutative? In other words, is the operation a commutative operation? Write your answer on the work sheet.

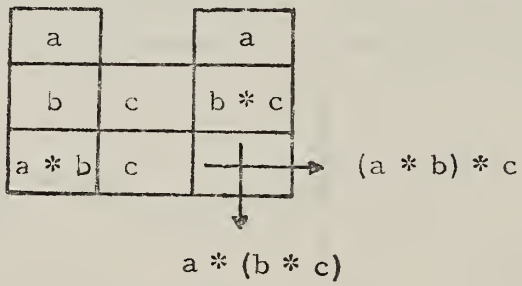
PART 109

A common treatment of the associative principle is to make it appear familiar by using an example like:

Add	
38	62
75	8
25	19

With the remark that the second number may be associated with either the first or the third.

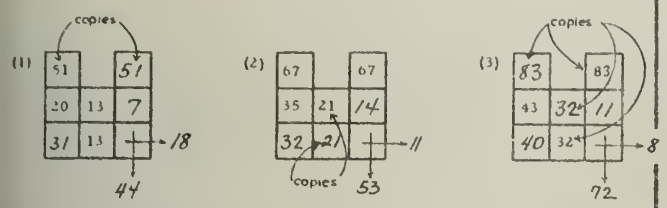
Such a treatment is not sufficiently precise to establish the pattern for the associative principle. Instead, the student is apt to establish something which is a consequence of the commutative and the associative principles. Rather than depend on such a vague treatment, we have developed a "cross-number" chart which maintains a unique pattern, regardless of the operation used in it.



After the pattern has been established for the operation subtraction, we have the student complete a chart for which ONE number appears in the last cell, regardless of the approach.

[Part 109] [Page 14]

Check your answers.



Record your results.

* * *

Here is the start of a chart-filling. There is something special about the numbers we have chosen. On your work sheet finish filling the cells in the chart in the manner used to complete the other charts. When you do this, you will discover what is special about the numbers we started with.

19		
7	0	
		? ?

[Part 109] [Page 15]

Check your answers.

19		19
7	0	7
12	0	12

Notice that only one number works for the lower-right-hand-corner cell. We get the same number when we do this:

19		
7		
12	0	?

as when we do this:

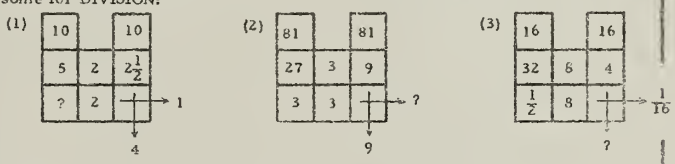
same number

		19
7	0	7
		?

Record your result.

* * *

We have been using chart-patterns like those above for SUBTRACTION. The same chart-pattern can also be used for other binary operations. Here are some for DIVISION:

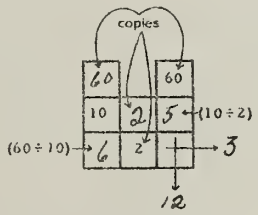


Answer the above questions on your work sheet.

After providing practice on cross-number charts for division, we again give the student a chart whose last cell is filled with only one number.

[Part 109] [Page 17]

Check your answers.

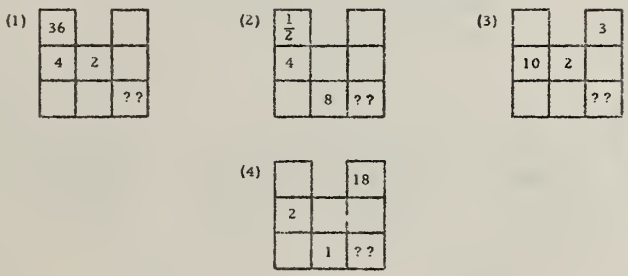


The two numbers that work are 3 and 12.

Record your results.

* * *

Now do these:



In the intervening pages the student has discovered that there is a restrictive condition under which, when he uses a prescribed pattern, the last cell in a division chart is filled with only one number.

Now he is given an ADDITION chart. He discovers the phenomenon that the pattern now will ALWAYS give only one number for the last cell.

[Part 109] [Page 21]

Check your answers.

(1)

9		9
3	1	3
3	1	3

(2)

97		97
68	1	68
$\frac{97}{68}$	1	$\frac{97}{68}$

Record your results.

* * *

We have used charts for the binary operations SUBTRACTION and DIVISION. We can use the same chart-pattern for the binary operation ADDITION. Here is the start of such an ADDITION chart.

2		
4	6	

We started with the numbers 2, 4, 6 so that exactly one number will work for the lower-right cell. This fact will enable you to check your computing as you fill the cells of the chart. Complete the chart on your work sheet.

[Part 109] [Page 22]

Check your answers.

The addition-chart starting with 2, 4, 6:

2		2
4	6	10
6	6	12

Exactly one number works for the lower-right cell.

Record your results.

* * *

Here are four more addition charts. For each, decide whether or not exactly one number will work for the lower-right cell.

MAKE SURE OF YOUR ANSWERS BY
COMPLETING THE CHARTS AND
DOING THE COMPUTATIONS CAREFULLY!

(1)

4		
6	8	

(2)

6		
8		
	10	

(3)

		8
10		
	12	

(4)

		20
22	24	

[Part 109] [Page 23]

Check your answers.

(1)

4		4
6	8	14
10	8	18

WORKS!

(2)

6		6
8	10	18
14	10	24

WORKS!

(3)

8		8
10	12	22
18	12	30

WORKS!

(4)

20		20
22	24	46
42	24	66

WORKS!

Record your results.

* * *

Here are some more addition charts. If all of these charts are properly completed, for how many of them will it turn out that only one number is needed for the lower-right cell?

10		
12		
	14	

10		
14		
	12	

		4
78	20	

		8
56	0	

		24
90		
	1	

		9
9		
	9	

$\frac{3}{8}$		
$\frac{1}{8}$	$\frac{2}{8}$	

1962		
1492		
	1776	

[Part 109] [Page 25]

Check your answer.

If you followed the patterns correctly, and if you didn't make any mistake in computing, you should have discovered that you couldn't make an addition chart that required two different numbers for the lower-right cell. In fact, nobody could make such an addition chart.

Record your result.

* * *

No matter what triple of numbers you pick, it won't matter whether you reach the lower-right cell like this:

4		
2		
6	7	13

or like this:

		4
2	7	9
		13

You must end up with the same number for the lower-right cell using either route.

Now, answer these questions on your work sheet.

(1)

		25
75	87	
		?

(2)

$\frac{3}{8}$		
$\frac{5}{11}$		
	$\frac{6}{11}$	?

Now, from the cross-number chart the pattern for the associative principle for addition is developed. Note the use of "half charts".

[Part 109] [Page 30]

Check your answers.

(1) (a)

136		
80		
	20	236

(b)

		136
80	20	
		236

(2)

687		637
12	8	20
	8	657

(3)

$7\frac{1}{2}$		$7\frac{1}{2}$
$\frac{5}{13}$	$\frac{8}{13}$	1
	$\frac{8}{13}$	$8\frac{1}{2}$

Record your results.

* * *

Here are two "halves" of the same addition chart. [Answer these questions on your work sheet.]

(1)

159		
75		
(159 + ?)	25	(159 + 75) + ?

(2)

		159
75	25	(75 + ?)
		? + (75 + 25)

(3) Look at your answers to problems (1) and (2) on your work sheet. Are the completed expressions in the lower-right cells equivalent?

Turn to PAGE 31.

[Part 109] [Page 31]

Check your answers.

Two "halves" of the same addition chart:

(1)

159		
75		
(159 + 75)	25	(159 + 75) + 25

(2)

		159
75	25	(75 + 25)
		159 + (75 + 25)

(3) Yes, these expressions are equivalent. We know this because we know that exactly one number will work for the lower-right cell in an addition chart. So, the expressions:

(159 + 75) + 25 and: 159 + (75 + 25)

must stand for the same number--that is, they are equivalent.

Record your results.

* * *

Here is an addition chart.

19		19
88	62	
	62	

And here is a filled-in left "half". Your job is to fill in the right "half".

19		
88		
(19 + 88)	62	(19 + 88) + 62

		19
88	62	(88 + ?)
		? + (88 + ?)

[Part 109] [Page 32]

Check your answers.

Addition Chart

Left half

19		
88		
(19 + 88)	62	(19 + 88) + 62

Right half

		19
88	62	(88 + 62)
		19 + (88 + 62)

equivalent

Record your results.

* * *

On your work sheet fill-in these addition chart "halves".

LEFT HALF

(1) (a)

14		
88		
(14 + 88)	92	(14 + 88) + 92

(2) (a)

17		
88		
88	76	(17 + 88) + 76

(3) (a)

	1
1	(10 + 88) + 66

RIGHT HALF

(b)

		14
88	92	(? + 92)
		? + (88 + ?)

(b)

		17
88	76	(? + ?)
		? + (? + 76)

(b)

	1
1	? + (? + ?)

[Part 109] [Page 35]

Check your answers.

(1) For the left "half" we associate the second number of the triple with the first number of the triple. (....., ___, ...) associated

(2) For the right "half" we associate the second number of the triple with the third number of the triple. (....., ___, ...) associated

(3)

	Left half		Right half
(a)	(78 + 8)	+ 2	78 + (8 + 2)
(b)	(127 + 36)	+ 64	127 + (36 + 64)
(c)	(283 + 570)	+ 77777	283 + (570 + 77777)
(d)	(126 + 126)	+ 126	126 + (126 + 126)

Consider yourself correct if you knew you could use another number for both place where 77777 is used.

Record your results.

* * *

You discovered earlier that, for our addition charts, whether you use the left-half pattern or the right-half pattern, you get the same number for the lower-right cell. This fact about the binary operation addition is expressed by what is called:

THE ASSOCIATIVE PRINCIPLE FOR ADDITION

[Pronounce 'associative' as 'a-SOSHEY-a-tive'.] Here is an instance of the associative principle for addition:

$$(81 + 79) + 21 = 81 + (79 + 21)$$

Simplify the expression on the left side of the equality sign, and then simplify the expression on the right side. Write your answers on your work sheet.

The first time the student meets the problem of proof is in the explanation of the exercise at the bottom of Page 40.

[Part 109] [Page 40]

Check your answers and record your results.

We can tell that the statement:

(3) (56 + 75) + 25 = 25 + (56 + 75)

is not an instance of the apa because a '25' follows the equality sign rather than a '56'. So, statement (3) does not fit a pattern-sentence of the associative principle for addition:

(___ + ...) + ,,,,,,, = ___ + (..... + ,,,,,,,)

However, statement (3) does fit a pattern-sentence of another principle:

___ + ... = ... + ___

Hence, we know that statement (3) is true without doing any computing. It is an instance of the commutative principle for addition.

* * *

Now, let us consider the last exercise of the set we have been discussing. Here is the statement, which you can see is true.

(4) $\underbrace{\underbrace{(75 + 93)}_{168} + 25}_{193} = \underbrace{93 + \underbrace{(75 + 25)}_{100}}_{193}$

But, it is not an instance of the apa:

(75 + ...) + ,,,,,,, = 93 + (..... + ,,,,,,,)

not copies

And, it is not an instance of the cpa:

(75 + 93) + ... = ... + (75 + 25)

not copies

However, you can use the apa and the cpa to see that (4) is a true statement.

Answer these questions.

(1) 75 + 93 = 93 + ?

(2) (75 + 93) + 25 = (93 + ?) + 25

(3) (93 + ?) + 25 = 93 + (75 + ?)

(4) (75 + 93) + 25 = 93 + (75 + ?)

Check your answers and record your results.

(1) 75 + 93 = 93 + 75

(2) (75 + 93) + 25 = (93 + 75) + 25

(3) (93 + 75) + 25 = 93 + (75 + 25)

(4) (75 + 93) + 25 = 93 + (75 + 25)

* * *

Look, again, at statement (4), above. Although this statement is an instance of neither the associative nor the commutative principle for addition, the statement is true. Also, it is possible to predict that statement (4) is true without doing any computing. Let's see how.

Fill the indicated blanks on your work sheet as you read the discussion.

We wish to conclude that

(75 + 93) + 25 = 93 + (75 + 25).

So, the first thing we might do is to notice that

(1) 75 + 93 = 93 + ___?___.

This follows from the ___?___ principle for ___?___. Now, since (75 + 93) is the same number as (93 + 75), if we add 25 to (75 + 93) we must get the same result as when we add ___?___ to (93 + 75). In other words,

(2) (75 + 93) + 25 = (93 + 75) + ___?___.

Also, it follows from the ___?___ principle for ___?___ that

(3) (93 + 75) + 25 = 93 + (___?___ + 25).

Now, from statement (2) we learn that

(75 + 93) + 25 is the same number as (93 + 75) + ___?___,

and from statement (3) we learn that

(93 + 75) + ___?___ is the same number as 93 + (___?___ + 25).

So, it must be the case that

(75 + 93) + 25 is the same number as 93 + (___?___ + 25).

In other words, it must be the case that

(4) (75 + 93) + 25 = 93 + (___?___ + 25).

Check your answers.

Here is a proof that

(75 + 93) + 25 = 93 + (75 + 25).

(1) 75 + 93 = 93 + 75 [commutative principle for addition]

(2) (75 + 93) + 25 = (93 + 75) + 25

(3) (93 + 75) + 25 = 93 + (75 + 25) [associative principle for addition.]

[From (2) we learn that (75 + 93) + 25 is the same number as (93 + 75) + 25, and from (3) we learn that (93 + 75) + 25 is the same number as 93 + (75 + 25). So, it must follows from (2) and (3) that (75 + 93) + 25 is the same number as 93 + (75 + 25). In other words:

(4) (75 + 93) + 25 = 93 + (75 + 25).

Record your results.

* * *

As we have seen, the statement

(75 + 93) + 25 = 93 + (75 + 25)

is an instance of neither the cpa nor the apa. But, our acceptance of this statement is logically based on our acceptance of both those principles.

Any statement which we accept because it is logically based on certain principles which we do accept is called a

CONSEQUENCE

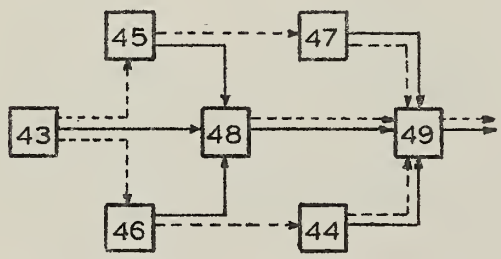
of those principles.

The statement:

(75 + 93) + 25 = 93 + (75 + 25)

is a consequence of the ___?___ principle for ___?___ and the ___?___ principle for ___?___. [Fill the indicated blanks on your work sheet.]

To test whether the student recognizes instances of the association principle, a branching sequence is provided. Here is the flow chart (solid arrows--correct response; dashed arrows--incorrect response):



[Part 109] [Page 43]

Check your answers.

The statement:

$$(75 + 93) + 25 = 93 + (75 + 25)$$

is a consequence of

the *commutative* principle for *addition*

and the *associative* principle for *addition*.

Record your results.

* * *

Look carefully at the following statement:

$$(75 + 35) + 35 = 35 + (35 + 75)$$

Answer these questions on your work sheet.

(1) Is the statement true?

(2) The statement is an instance of ___?___.

(a) the apa (b) the cpa (c) neither the apa nor the cpa

If your answer to (2) is (a), turn to PAGE 45.
If your answer to (2) is (b), turn to PAGE 46.
If your answer to (2) is (c), turn to PAGE 48.

[Part 109] [Page 44]

Check your answer from page 46.

Look at the given statement:

$$(75 + 35) + 35 = 35 + (35 + 75)$$

and look at a pattern-sentence for the apa:

$$(___ + \ldots\ldots) + \ldots\ldots = ___ + (\ldots\ldots + \ldots\ldots)$$

The given statement has a '75' here but it doesn't have a '75' here.

Record your results.

* * *

Convert each of the following into a true statement by filling the blank with a copy of one of the following numerals:

3, 4, 5, 6, 7, 10, (8 + 19), (6 + 6), (19 + 8)

[Copy the numeral in the blank on your work sheet.]

(1) (4 + 25) + 8 = ___?___ + (25 + 8)
(2) (9 + 37) + 5 = ___?___ + (37 + 9)
(3) (3 + 3) + 7 = ___?___ + (7 + 3)
(4) (8 + 19) + 19 = 19 + ___?___
(5) (3 + 4) + 12 = 12 + ___?___
(6) (5 + 10) + 5 = 5 + (5 + ___?___)
(7) (6 + 6) + 6 = ___?___ + (6 + 6)

Now, for each exercise, tell whether your completed statement is an instance of the apa, the cpa, or of neither of these principles. [Indicate this by writing an 'apa', a 'cpa', or a 'neither' in the second blank on your work sheet.]

Turn to PAGE 49.

[Part 109] [Page 45]

Check your answers from page 43.

(1) *Yes*, the statement:

$$(75 + 35) + 35 = 35 + (35 + 75)$$

is true.

(2) You said that the statement is an instance of the apa.

All you have to do is compare the given sentence with the pattern-sentence:

$$(75 + 35) + 35 = 35 + (35 + 75)$$

not copies

$$(___ + \ldots\ldots) + \ldots\ldots = ___ + (\ldots\ldots + \ldots\ldots)$$

Record your results.

* * *

[Your answer to the following question should be written in the Page 45 section of your work sheet.]

Here is the statement again:

$$(75 + 35) + 35 = 35 + (35 + 75)$$

Is it an instance of the commutative principle for addition?

If you think it is, write a 'yes' on your work sheet, and turn to PAGE 47.

If you think now that it is not an instance of either the apa or the cpa, write a 'no' on your work sheet and turn to PAGE 48.

[Part 109] [Page 46]

Check your answers from page 43.

(1) *Yes*, the statement

$$(75 + 35) + 35 = 35 + (35 + 75)$$

is true.

(2) You said that this statement is an instance of the cpa.

After looking at so many instances of the cpa in an earlier part, you should recognize immediately that the statement:

$$(75 + 35) + 35 = 75 + (35 + 75)$$

not copies

doesn't fit the pattern-sentence:

$$___ + \ldots\ldots = \ldots\ldots + ___$$

[Of course, the given statement is logically based on the cpa. So, it is a consequence of the cpa. But, it is not an instance just because it doesn't fit the pattern.]

Record your results.

* * *

[Your answer to the following question should be written in the Page 46 section of your work sheet.]

Here is the statement again:

$$(75 + 35) + 35 = 35 + (35 + 75)$$

Is it an instance of the associative principle for addition?

If you think it is, write a 'yes' on your work sheet, and turn to PAGE 44.

If you think now that it is not an instance of either the cpa or the apa, write a 'no' on your work sheet and turn to PAGE 48.

[Part 109] [Page 47]

Check your answer from page 45.

Look at the given statement:

$$(75 + 35) + 35 = 35 + (35 + 75)$$

and look at a pattern-sentence for the cpa:

$$___ + \ldots\ldots = \ldots\ldots + ___$$

The given statement has a '(75 + 35)' here but it doesn't have a copy of '(75 + 35)' here.

Record your result.

* * *

Convert each of the following into a true statement by filling the blank with a copy of one of the following numerals:

3, 4, 5, 6, 7, 10, (8 + 19), (6 + 6), (19 + 8)

[Copy the numeral in the blank on your work sheet.]

(1) (4 + 25) + 8 = ? + (25 + 8)
(2) (9 + 37) + 5 = ? + (37 + 9)
(3) (3 + 3) + 7 = ? + (7 + 3)
(4) (8 + 19) + 19 = 19 + ?
(5) (3 + 4) + 12 = 12 + ?
(6) (5 + 10) + 5 = 5 + (5 + ?)
(7) (6 + 6) + 6 = ? + (6 + 6)

Now, tell whether your completed statement is an instance of the apa, the cpa, or of neither of these principles. [Indicate this by writing an 'apa', a 'cpa', or a 'neither' in the second blank on your work sheet.]

Turn to PAGE 49.

[Part 109] [Page 48]

Check your answers.

(1) *Yes*, the given statement:

$$(75 + 35) + 35 = 35 + (35 + 75)$$

is true, but

(2) it is an instance of *neither the apa nor the cpa*.

Record your results.

* * *

Convert each of the following into a true statement by filling the blank with a copy of one of the following numerals:

3, 4, 5, 6, 7, 10, (8 + 19), (6 + 6), (19 + 8)

[Copy the numeral in the blank on your work sheet.]

(1) (4 + 25) + 8 = ? + (25 + 8)
(2) (9 + 37) + 5 = ? + (37 + 9)
(3) (3 + 3) + 7 = ? + (7 + 3)
(4) (8 + 19) + 19 = 19 + ?
(5) (3 + 4) + 12 = 12 + ?
(6) (5 + 10) + 5 = 5 + (5 + ?)
(7) (6 + 6) + 6 = ? + (6 + 6)

Now, tell whether your completed statement is an instance of the apa, the cpa, or of neither of the principles. [Indicate this by writing an 'apa', a 'cpa', or a 'neither' in the second blank on your work sheet.]

[Part 109] [Page 49]

Check your answers.

	Is an instance of
(1) (4 + 25) + 8 = *4* + (25 + 8)	[*apa*]
(2) (9 + 37) + 5 = *5* + (37 + 9)	*neither* the apa nor the cpa
(3) (3 + 3) + 7 = *3* + (7 + 3)	*neither* the apa nor the cpa
(4) (8 + 19) + 19 = 19 + *(8+19)*	[*cpa*]
or: (8 + 19) + 19 = 19 + *(19+8)*	*neither* the apa nor the cpa
(5) (3 + 4) + 12 = 12 + *7*	*neither* the apa nor the cpa
(6) (5 + 10) + 5 = 5 + (5 + *10*)	[*cpa*]
(7) (6 + 6) + 6 = *6* + (6 + 6)	[*apa*] or: [*cpa*]

Record your results.

* * *

Put a numeral in each blank on your work sheet to make a statement which is an instance of the associative principle for addition.

(1) (3 + 5) + 2 = ? + (5 + 2)
(2) (7 + 2) + (4 + 3) = ? + [2 + (4 + 3)]
(3) [(2 + 9) + 6] + 5 = ? + (6 + 5)
(4) [(2 + 9) + 6] + 5 = (2 + 9) + (? + 5)
(5) (11 + 6) + (4 + 1) = ? + [6 + (4 + 1)]
(6) (15 + 3) + (8 + 2) = 15 + [3 + (?)]

Turn to PAGE 50.

The students have been shown an example of some principles of logic which they can use in proofs of mathematical statements. They are now asked to fill in the blanks of a proof. Finally, they are asked to construct a proof. [Page 70].

[Part 109] [Page 65]

Check your answers.

Here is a proof that

$$(493 + 39) + 7 = (493 + 7) + 39.$$

$(493 + 39) + 7 = (39 + \underline{493}) + 7$ [cpa]

$(39 + 493) + 7 = 39 + (\underline{493} + 7)$ [apa]

$39 + (493 + 7) = (493 + \underline{7}) + 39$ [cpa]

Therefore,

$$(493 + 39) + 7 = (493 + \underline{7}) + 39.$$

[This conclusion is a consequence of the cpa and the apa.]

Of course, anyone who accepts this conclusion can simplify '493 + 39 + 7' in a clever way. An abbreviation of the conclusion is:

$$493 + 39 + 7 = \underbrace{\underbrace{493 + 7}_{500} + 39}_{539}$$

So, '493 + 39 + 7' simplifies to:

Record your results.

* * *

On your work sheet complete the following proof of the statement:

$$(25 \times 46) \times 4 = 46 \times (25 \times 4)$$

$(25 \times 46) \times 4 = (? \times ?) \times 4$ [cpm]

$(? \times ?) \times 4 = 46 \times (? \times 4)$ [?]

Therefore,

$$(25 \times 46) \times 4 = ? \times (? \times ?).$$

[This conclusion is a consequence of the ? and the ?.]

[Part 109] [Page 66]

Check your answers.

A proof that

$$(25 \times 46) \times 4 = 46 \times (25 \times 4).$$

$(25 \times 46) \times 4 = (\underline{46} \times \underline{25}) \times 4$ [cpm]

$(\underline{46} \times \underline{25}) \times 4 = 46 \times (\underline{25} \times 4)$ [apm]

Therefore,

$$(25 \times 46) \times 4 = \underline{46} \times (\underline{25} \times \underline{4}).$$

[This conclusion is a consequence of the cpm and the apm.]

Notice again that, based on this conclusion, '25 × 46 × 4' simplifies easily to '4600'.

Record your results.

* * *

On your work sheet, write a proof of the statement:

$$(88 + 76) + 12 = (88 + 12) + 76$$

[If you need more space than is provided on the work sheet, you may put your proof on the back of the work sheet page.]

[Part 109] [Page 67]

Check your proof.

Here are TWO proofs of the statement:

$$88 + 76 + 12 = 88 + 12 + 76$$

[Notice that we have used our convention for omitting grouping symbols.]

Proof I

$88 + 76 + 12 = 88 + (76 + 12)$ [apa]

$88 + (76 + 12) = 88 + (12 + 76)$ [cpa]

$88 + (12 + 76) = 88 + 12 + 76$ [apa]

Therefore,

$$88 + 76 + 12 = 88 + 12 + 76.$$

[This conclusion is a consequence of the apa and the cpa.]

Proof II

$88 + 76 + 12 = 12 + (88 + 76)$ [cpa]

$12 + (88 + 76) = 12 + 88 + 76$ [apa]

$12 + 88 + 76 = 88 + 12 + 76$ [cpa]

Therefore,

$$88 + 76 + 12 = 88 + 12 + 76.$$

[This conclusion is a consequence of the apa and the cpa.]

Record your results. If you wrote either of these two proofs, circle the 'OK' on the work sheet. If you wrote a different proof and you think it is correct, circle the 'OK' also. Later, as you gain experience with proofs, you will usually know if your proof is correct.

[Part 109] [Page 69]

Check your answers.

Expression to be simplified	Simplify this expression instead
(17 + 25) + (75 + 3)	(25 + 75) + (17 + 3)
42 78	100 20
120	120

Record your results.

* * *

Now, in the example above, it is certainly easier to simplify the expression on the right than to simplify the given expression. But, when you pick the easier one to simplify instead, you do so because you believe that the two expressions are equivalent. In other words, you believe that

(☆) (17 + 25) + (75 + 3) = (25 + 75) + (17 + 3).

Let's see what is the basis of this belief. That is, let's prove (☆). [Fill the blanks on your work sheet.]

(17 + 25) + (75 + 3) = [(17 + 25) + 7] + 3 [apa]

[(17 + 25) + 75] + 3 = [17 + (7 + 75)] + 3 [?]

[17 + (? + 75)] + 3 = [(? + 75) + 7] + 3 [cpa]

[(? + 75) + 17] + 3 = (? + 75) + (? + 3) [apa]

Therefore,

(17 + 25) + (75 + 3) = (? + 75) + (? + 3).

[So, the basis of this conclusion is the ___?___ and the ___?___.]

[Part 109] [Page 70]

Check your answers.

A proof of the statement '(17 + 25) + (75 + 3) = (25 + 75) + (17 + 3)'

(17 + 25) + (75 + 3) = [(17 + 25) + 75] + 3 [apa]

[(17 + 25) + 75] + 3 = [17 + (25 + 75)] + 3 [~~apa~~]

[17 + (25 + 75)] + 3 = [(25 + 75) + 17] + 3 [cpa]

[(25 + 75) + 17] + 3 = (25 + 75) + (17 + 3) [apa]

Therefore,

(17 + 25) + (75 + 3) = (25 + 75) + (17 + 3).

[So, the basis of this conclusion is the apa and the cpa.]

Record your results.

* * *

Now, you write a proof [on your work sheet] of the statement:

$$(25 \times 33\tfrac{1}{3}) \times (3 \times 4) = (33\tfrac{1}{3} \times 3) \times (25 \times 4)$$

Try to give a proof that has no more lines in it than the proof above.

We now present the students with a "new" binary operation, ☆, which is defined by:

$$\forall_a \forall_b \quad a \star b = \frac{a+b}{2}$$

We call this operation:

averaging

The students are asked to investigate the operation to see whether or not it is an associative operation.

[Part 109] [Page 89]

Check your answers.

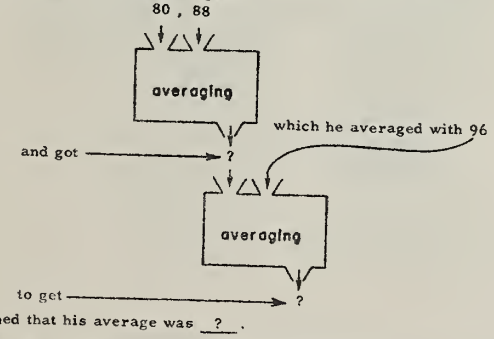
Simplify:

		Answer:
(1) $50 \times (2 \times 5687)$	$[= 5687 \times (50 \times 2)]$	568700
(2) $(97 + 107) + (3 + 93)$	$[= (97 + 3) + (107 + 93)]$	300
(3) $(4 \times 1492) \times 0.25$	$[= 1492 \times (0.25 \times 4)]$	1492
(4) $(7 \times 8\frac{1}{3}) \times (6 \times 12)$	$[= (7 \times 6) \times (8\frac{1}{3} \times 12)]$	4200
(5) $(11 \times 33\frac{1}{3}) \times (3 \times 5)$	$[= (11 \times 5) \times (33\frac{1}{3} \times 3)]$	5500
(6) $(8 \times 9) \times (\frac{7}{8} \times \frac{2}{3})$	$[= (\frac{7}{8} \times 8) \times (\frac{2}{3} \times 9)]$	42

Record your results.

* * *

Sammy took three math tests during the first 6 weeks of his course. On these tests he earned 80, 88, and 96. Sammy wished to compute his average for the 6-weeks period, and he thought he knew an easy way to figure it out. He simply got the average of his scores by taking two numbers at a time. First he averaged



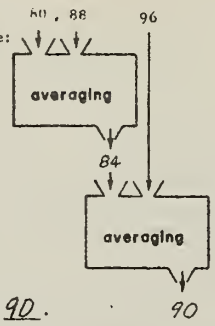
So, he claimed that his average was ___?___.

[Answer the three indicated questions on your work sheet.]

[Part 109] [Page 90]

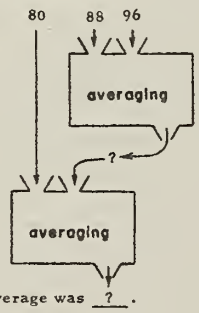
Check your answers and record your results.

This is how Sammy computed his average:



So, Sammy claimed that his average was 90 .

Milton got the same scores as Sammy did, and this is how Milton computed his average:

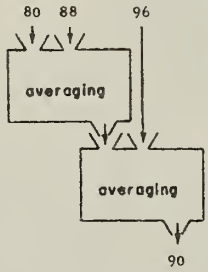


Milton claimed that his average was ? .

[Part 109] [Page 91]

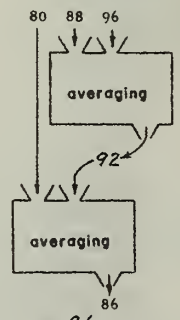
Check your answers and record your results.

Sammy's way of averaging 3 scores.



Sammy claimed 90 for his average.

Milton's way of averaging 3 scores.



Milton claimed 86 for his average.

Both Sammy and Milton were surprised that they came out with different results. Fred wasn't surprised at all. Fred didn't accept the ? principle for averaging a pair of numbers.

[Answer this question on your work sheet.]

[Part 109] [Page 92]

Check your answers.

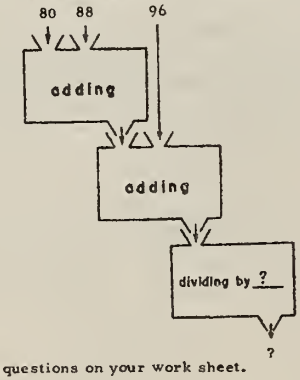
Sammy and Milton knew that if you wished to find the "sum" of three numbers, you could get it by adding two at a time. They thought that this idea worked for averaging also. Of course, if the idea did work, they should have come out with the same result. But they didn't get the same result. So, the idea didn't work for averaging.

Fred knew the idea wouldn't work because he knew that averaging two numbers is not an associative operation.

Record your result.

* * *

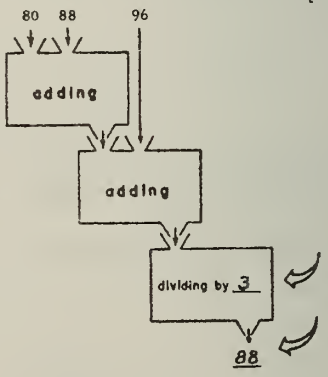
Fred drew this picture to show Sammy and Milton how to find their average:



Answer these two questions on your work sheet.

[Part 109] [Page 93]

Check your answers.



Record your results.

* * *

Fred did not have to explain to anybody how to get the average of his scores. They were:

100 100 100

* * *

This is the end of Part 109. Insert your work sheet under the front cover of this booklet and return them to your teacher.

PART 110

Although students of algebra have always accepted statements such as

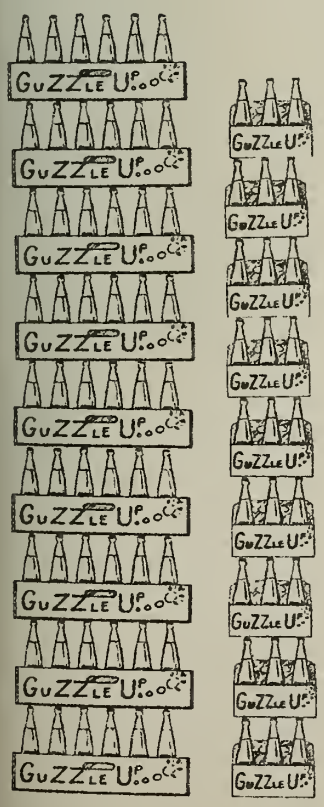
$$'3x + 7x = 10x',$$

they are not nearly as ready to accept

$$'(3 \times 19) + (7 \times 19) = 10 \times 19'.$$

This is evidence that algebra students do not have a mathematical rationale for one of their most valuable mathematical skills. It is traditional with UICSM to develop within the student a strong conviction of the distributive principle of multiplication over addition. On the following pages are excerpts from this development in the programed materials.

[Part 110] [Page 1]



The Zabranchburg sophomores are planning a class picnic.

They have nine 24-bottle cases of pop and nine 6-bottle cartons of pop.

How many bottles of pop do they have altogether?

Turn to PAGE 2.

[Part 110] [Page 5]

Check your answer.

$(45 \times 27) + (55 \times 27) = 2700$ $27.00

Record your results.

* * *

Sammy's father gives him 75 cents each time he mows the lawn and 25 cents each time he washes the car. Last summer Sammy mowed the lawn 13 times and washed the car 13 times. How much did Sammy earn for doing this?

[Part 110] [Page 6]

Check your answer.

$(75 \times 13) + (25 \times 13) = 1300$ $13.00

Record your result.

* * *

Answer these questions on your work sheet.

(1) At 75¢ per lawn-mowing and 25¢ per car-wash, how much would Sammy earn in a day during which he mowed the lawn and washed the car, both?

(2) If Sammy worked 13 such double-job days, how much would he earn altogether?

(3) (a) How many times would the car get washed by Sammy during those 13 days?

(b) How many times would the lawn get mowed by Sammy during those 13 days?

[Part 110] [Page 9]

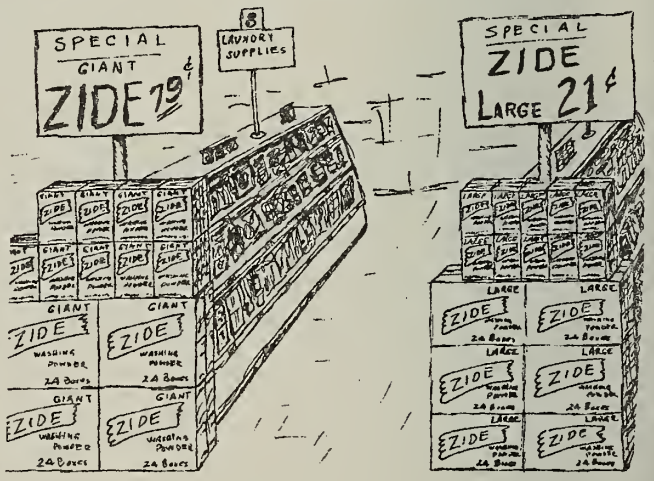
Check your answer.

$100 \times 37 = 3700$ $37.00

Record your result.

* * *

The Zabranchburg Zupermarket was selling Zide detergent at practically give-away prices. A giant-size box of Zide was priced at 79 cents, and a large-size box was only 21 cents. So many housewives took advantage of this once-in-a-lifetime sale that the store soon sold all their dozen gross of giant-size boxes and all their dozen gross of large boxes. How much money did the store collect just from their sale of Zide? [1 dozen gross = 1728]



[Part 110] [Page 10]

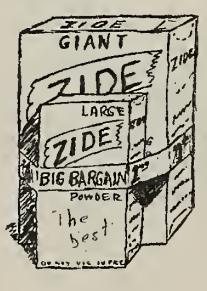
Check your answer.

$(79 \times 1728) + (21 \times 1728) = 172800$ $1728.00

Record your result.

* * *

The Zide sale had been so successful that the manager ordered another dozen gross giant-size boxes and another dozen gross large-size boxes. This time, however, the store taped a large box to each giant box, and advertised both for one dollar. How much money would they collect this time if they sold all 1728 pairs of Zide boxes?



[Part 110] [Page 16]

Check your answer.

$(25 \times 7) + (75 \times 7) =$ 700

Record your result.

* * *

Sally's mother sent her to the post office to buy twenty-eight 7-cent air-mail stamps and twenty-eight 3-cent stamps. She gave Sally 9 nickels and 9 quarters to pay for the stamps. Sally told her mother that she would need more money to pay for all those stamps. How much more?

(a) a penny (b) a nickel (c) a dime (d) a quarter

(e) a half-dollar (f) a silver dollar

Check your answer.

$(57 \times 4) + (43 \times 4) = 400$ 400 miles

Record your result.

* * *

Answer these questions on your work sheet.

(1) (a) $300 + 400 =$?

(b) $(3 \times 100) + (4 \times 100) =$? $\times 100$

(2) (a) $\frac{1}{4} + \frac{2}{4} =$?

(b) $(1 \times \frac{1}{4}) + (2 \times \frac{1}{4}) =$? $\times \frac{1}{4}$

(3) (a) $0.2 + 0.5 =$?

(b) $(2 \times 0.1) + (5 \times 0.1) =$? $\times 0.1$

(4) (a) $63 + 37 =$?

(b) $(63 \times 1) + (37 \times 1) =$? $\times 1$

Check your answers.

(1) (a) $300 + 400 =$ 700

(b) $(3 \times 100) + (4 \times 100) =$ 7 $\times 100$

(2) (a) $\frac{1}{4} + \frac{2}{4} =$ 3/4

(b) $(1 \times \frac{1}{4}) + (2 \times \frac{1}{4}) =$ 3 $\times \frac{1}{4}$

(3) (a) $0.2 + 0.5 =$ 0.7

(b) $(2 \times 0.1) + (5 \times 0.1) =$ 7 $\times 0.1$

(4) (a) $63 + 37 =$ 100

(b) $(63 \times 1) + (37 \times 1) =$ 100 $\times 1$

Record your results.

* * *

Give the whole numbers which divide 12. ['divide' means the same thing as 'divide exactly'.]

Answers: 1, 2, 3, 4, 6, 12

Give the whole numbers other than 1 and 8 which divide 8.

Answers: 2, 4

For each number listed below, give just one whole number which divides it. Don't give 1 or the listed number for your divisors.

(1) 6 (2) 10 (3) 9 (4) 25

Check your answers.

	Given number	Divisors
(1)	26	2, 13
(2)	15	3, 5
(3)	21	3, 7
(4)	13 + 14 [= 27]	3, 9
(5)	2 × 11 [= 22]	2, 11

Record your results.

* * *

Give two divisors of each number listed. [Don't give 1 or the number itself.]

(1) 3×6 (2) $(4 \times 8) + 1$ (3) $(3 \times 5) + (2 \times 3)$

Check your answers.

	Given number	Two divisors
(1)	35	7, 5
(2)	55	5, 11
(3)	77	11, 7

Record your results.

* * *

Give two divisors of each listed number. [Don't give 1 or the number itself.]

(1) $(3 \times 5) + (4 \times 5)$

(2) $(2 \times 11) + (3 \times 11)$

(3) $(2 \times 7) + (9 \times 7)$

(4) $(13 \times 5) + (4 \times 5)$

Check your answers.

Given number	Two divisors
(1) $20 \times 2 + 9 \times 2$	29, 2
(2) $19 \times 3 + 10 \times 3$	29, 3
(3) $18 \times 23 + 11 \times 23$	29, 23
(4) $29 \times 37 + 0 \times 37$	29, 37

Record your results.

* * *

Complete each of the following sentences with whole numbers other than 1.

Sample. $\underbrace{\underbrace{8 \times 3}_{24} + \underbrace{9 \times 3}_{27}}_{51} = _?_ \times _?_$

Solution. $8 \times 3 + 9 \times 3 = _17_ \times _3_$

(1) $20 \times 2 + 9 \times 2 = _?_ \times _?_$

(2) $19 \times 3 + 10 \times 3 = _?_ \times _?_$

(3) $18 \times 23 + 11 \times 23 = _?_ \times _?_$

(4) $29 \times 37 + 0 \times 37 = _?_ \times _?_$

Check your answers.

(1) $3 \times 7 + 2 \times 7 = _5_ \times _7_$

(2) $4 \times 11 + 1 \times 11 = _5_ \times _11_$

(3) $10 \times 17 + 9 \times 17 = 19 \times _17_$

(4) $34 \times 43 + 13 \times 43 = _47_ \times _43_$

Record your results.

* * *

Complete each of the following sentences with whole numbers other than 1.

(1) $50 \times 313 + 47 \times 313 = _?_ \times _?_$

(2) $79 \times 997 + 4 \times 997 = _?_ \times _?_$

(3) $900 \times 211 + 91 \times 211 = _?_ \times _?_$

(4) $555 \times 919 + 22 \times 919 = _?_ \times _?_$

(5) $727 \times 727 + 70 \times 727 = _?_ \times _?_$

(6) $244 \times 599 + 255 \times 599 = _?_ \times _?_$

(7) $1 \times 911 + 1 \times 911 = _?_ \times _?_$

(8) $2 \times 911 + 0 \times 911 = _?_ \times _?_$

Check your answers.

(1) $\underbrace{\underbrace{4 \times 11}_{44} + \underbrace{3 \times 11}_{33}}_{77} = \underbrace{(\underbrace{4 + 3}_{7}) \times 11}_{7 \times ?}$

(2) $2 \times 11 + 2 \times 11 = (2 + \underline{2}) \times 11$

(3) $2 \times 13 + 8 \times 13 = (\underline{2} + 8) \times 13$

(4) $1900 \times 1776 + 62 \times 1776 = (1900 + \underline{62}) \times 1776$

Record your results.

* * *

Simplify:

(1) $3 \times 8 + 7 \times 8$

(2) $6 \times 9 + 4 \times 9$

(3) $8 \times 17 + 2 \times 17$

(4) $75 \times 1776 + 25 \times 1776$

Check your answers.

(1) $\frac{1}{4} \times 8 + \frac{3}{4} \times 8 = \underline{8}$

(2) $\frac{1}{4} \times 36 + \frac{3}{4} \times 36 = \underline{36}$

(3) $\frac{1}{4} \times 144 + \frac{3}{4} \times 144 = \underline{144}$

(4) $\frac{1}{4} \times 7 + \frac{3}{4} \times 7 = \underline{7}$

(5) $\frac{1}{4} \times 11 + \frac{3}{4} \times 11 = \underline{11}$

Record your results.

* * *

Simplify:

(1) $\frac{2}{3} \times 12 + \frac{1}{3} \times 12$

(2) $\frac{2}{3} \times 33 + \frac{1}{3} \times 33$

(3) $\frac{2}{3} \times 1962 + \frac{1}{3} \times 1962$

(4) $\frac{3}{8} \times 888 + \frac{5}{8} \times 888$

(5) $\frac{3}{8} \times 11 + \frac{5}{8} \times 11$

(6) $\frac{3}{8} \times 1776 + \frac{5}{8} \times 1776$

(7) $\frac{3}{17} \times 1492 + \frac{14}{17} \times 1492$

(8) $\frac{31}{65} \times 123456789 + \frac{34}{65} \times 123456789$

(9) $\frac{6}{34} \times 2468 + \frac{11}{34} \times 2468$

To further emphasize the pattern for the distributive principle, the "machine" device is again presented.

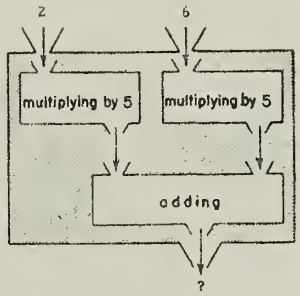
[Part 110] [Page 37]

Check your answer.

Record your result.

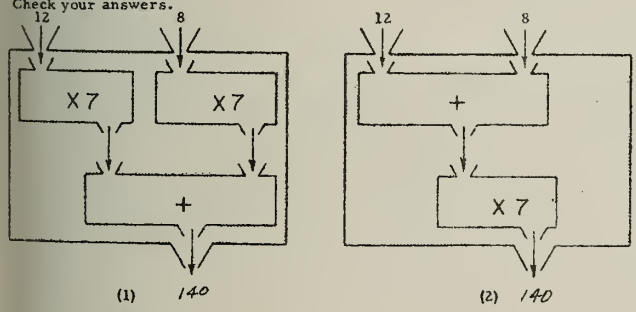
* * *

Now, answer this question.



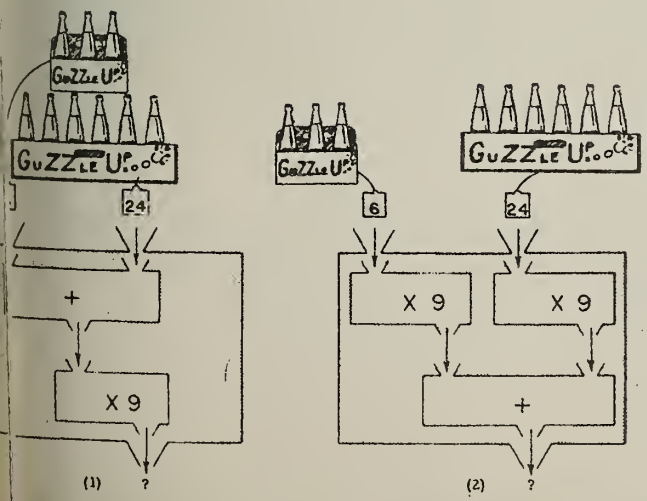
[Part 110] [Page 39]

Check your answers.



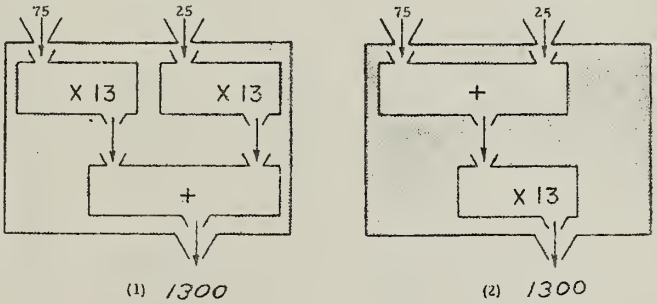
Record your results.

* * *



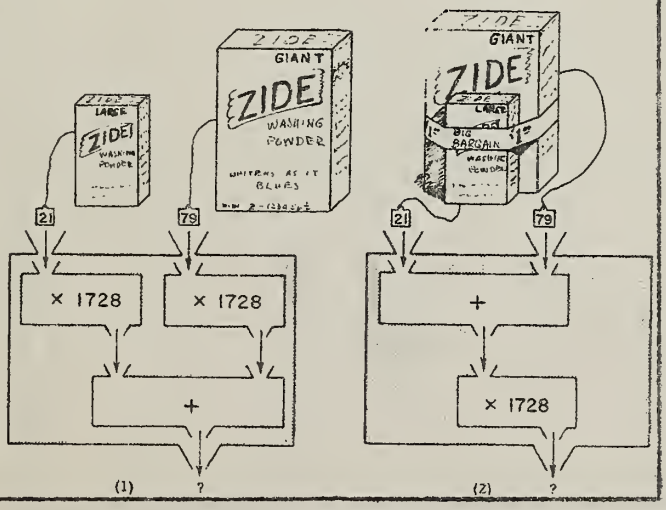
[Part 110] [Page 42]

Check your answers.



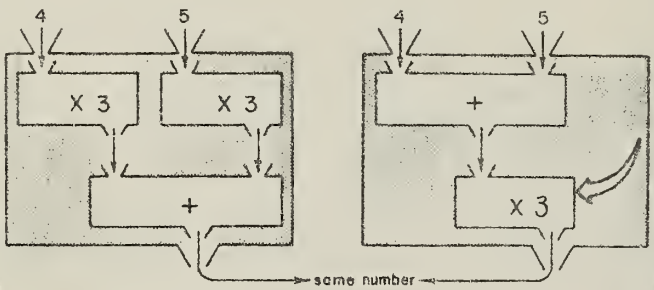
Record your results.

* * *



[Part 110] [Page 46]

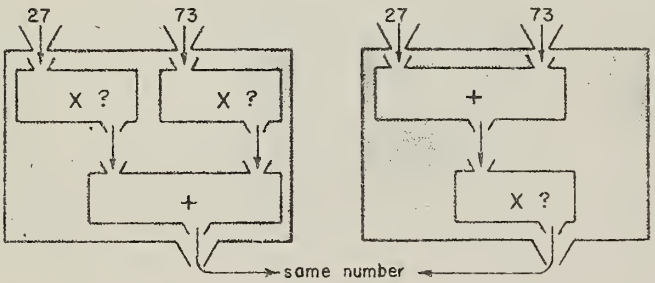
Check your answer.



Record your result.

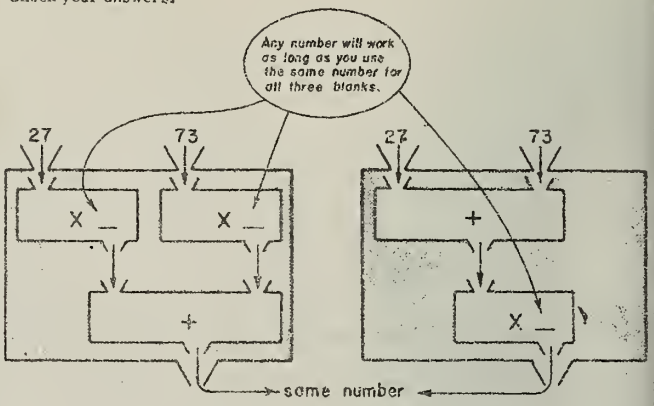
* * *

Answer the three questions on your work sheet.



[Part 110] [Page 47]

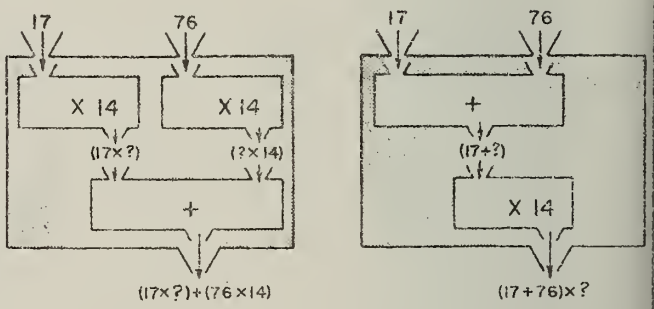
Check your answers.



Record your results.

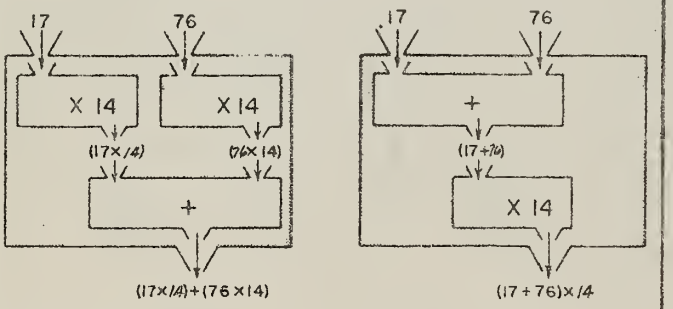
* * *

Answer these questions.



[Part 110] [Page 48]

Check your answers.



Record your results.

* * *

True or false?

$$(17 \times 14) + (76 \times 14) = (17 + 76) \times 14$$

Circle the 'True' or the 'False' on your work sheet.

Practice in writing and recognizing instances of the distributive principle is provided in various forms.

[Part 110] [Page 53]

Check your answer.

$(43 \times 91) + (29 \times 91) = (43 + 29) \times 91$

Record your result.

* * *

You probably accepted the statements:

$(44 \times 999) + (55 \times 999) = (44 + 55) \times 999$

$(81 \times 52) + (19 \times 52) = (81 + 19) \times 52$

$(43 \times 91) + (29 \times 91) = (43 + 29) \times 91$

without doing any computation. You were right in doing so.

True or False:

(1) $(17 \times 932) + (19 \times 932) = (17 + 19) \times 932$

(2) $(17 \times 932) + (17 \times 932) = (17 + 17) \times 932$

(3) $(17 \times 17) + (17 \times 17) = (17 + 17) \times 17$

(4) $(63 \times 63) + (63 \times 63) = (63 + 63) \times 2$

[Circle the 'True' or the 'False' on your work sheet to indicate your answer to each problem.]

[Part 110] [Page 54]

Check your answers.

(1) $(17 \times 932) + (19 \times 932) = (17 + 19) \times 932$ True

(2) $(17 \times 932) + (17 \times 932) = (17 + 17) \times 932$ True

(3) $(17 \times 17) + (17 \times 17) = (17 + 17) \times 17$ True

(4) $(63 \times 63) + (63 \times 63) = (63 + 63) \times 2$ False

Record your results.

* * *

Look at this true statement:

$(17 \times 932) + (19 \times 932) = (17 + 19) \times 932$

This statement as well as many others you have looked at so far in this part have the same pattern as the sentence:

$(___ \times \ldots\ldots) + (___ \times \ldots\ldots) = (___ + ___) \times \ldots\ldots$

Each statement that fits this pattern-sentence is an instance of a principle about both multiplication <u>and</u> addition of numbers of arithmetic. This principle is called:

THE DISTRIBUTIVE PRINCIPLE FOR

MULTIPLICATION OVER ____?____.

Fill the blank on your work sheet.

[Part 110] [Page 57]

Check your answers.

Here is one of the instances of the distributive principles for multiplication over addition:

indicated sum indicated sum

$(75 \times 85) + (34 \times 85) = (75 + 34) \times 85$

indicated products indicated product

(1) There are 3 multiplication signs in the instance above.

(2) There are 3 indicated products in the instance above.

(3) There are 2 addition signs in the instance above.

(4) There are 2 indicated sums in the instance above.

(5) '$\times$' is the principal operator of '$(75 + 34) \times 85$'.

(6) '$+$' is the principal operator of '$(75 \times 85) + (34 \times 85)$'.

Record your results.

* * *

Here is the beginning of an instance of the distributive principle for multiplication over addition:

$(93 \times 58) + (32 \times$

Complete it on your work sheet.

[Part 110] [Page 59]

Check your answers.

<u>Instances of the d-p-m-over-a</u>

(1) $(19 \times 72) + (31 \times 72) = (19 + 31) \times 72$

(2) $(8 \times 16) + (43 \times 16) = (8 + 43) \times 16$

(3) $(25 \times 20) + (20 \times 20) = (25 + 20) \times 20$

(4) $(37 \times 37) + (9 \times 37) = (37 + 9) \times 37$

or, by using our convention for omitting grouping symbols:

$37 \times 37 + 9 \times 37 = (37 + 9) \times 37$

Record your results.

* * *

Again, a pattern-sentence for the distributive principle for multiplication over addition is:

$(___ \times \ldots\ldots) + (___ \times \ldots\ldots) = (___ + ___) \times \ldots\ldots$

On your work sheet, write this pattern-sentence, omitting as many grouping symbols as possible according to our convention, as in Exercise (4) above.

Check your answer.

A pattern-sentence for the dpma with no grouping symbols omitted is:

(___ × .....) + (___ × .....) = (___ + ___) × .....

but when it is written with as many grouping symbols omitted as possible according to the convention it looks like this:

___ × ..... + ___ × ..... = (___ + ___) × .....

Record your result.

* * *

Exactly one of the statements below is not an instance of the dpma. Which one? [Circle your answer on your work sheet.]

(1) 25 × 14 + 75 × 14 = (25 + 75) × 14

(2) (25 × 14) + (75 × 14) = (25 + 75) × 14

(3) 25 × (7 × 2) + 75 × (7 × 2) = (25 + 75) × (7 × 2)

(4) 25 × (7 + 7) + (25 + 50) × (7 + 7) = [25 + (25 + 50)] × (7 + 7)

(5) (5 × 5) × 14 + (50 + 25) × 14 = (5 × 5) + (50 + 25) × 14

(6) (5 × 5) × 14 + 75 × 14 = [(5 × 5) + 75] × 14

Check your answers.

(1) 9 × 8 + 11 × 8 = (9 + 11) × 8 (copies)

(2) 3 × 18 + 7 × 18 = (3 + 7) × 18

(3) 8 × 37 + 2 × 37 = (8 + 2) × 37

Record your results.

* * *

Fill in the blanks on your work sheet so that the completed sentences are true, and are also instances of the distributive principle for multiplication over addition.

(1) 75 × ..... + ___ × ..... = (___ + 25) × 84

(2) 37 × ..... + 63 × 61 = (___ + ___) × .....

(3) ___ × 73 + ___ × ..... = (65 + 35) × .....

Check your answers.

(1) 75 × 84 + 25 × 84 = (75 + 25) × 84

(2) 37 × 61 + 63 × 61 = (37 + 63) × 61

(3) 65 × 73 + 35 × 73 = (65 + 35) × 73

Record your results.

* * *

For each exercise below, you have been given one side of an instance of the distributive principle for multiplication over addition. Complete each instance on your work sheet.

(1) 3 × π + 4 × π = 7

(2) ? = (3 + 4) × 5

(3) 7 × $\frac{3}{8}$ + 1 × $\frac{3}{8}$ = 7

(4) ? = ($\frac{2}{7}$ + $\frac{5}{7}$) × π

Check your answers.

(1) 3 × π + 4 × π = (3 + 4) × π

(2) 3 × 5 + 4 × 5 = (3 + 4) × 5

(3) 7 × $\frac{3}{8}$ + 1 × $\frac{3}{8}$ = (7 + 1) × $\frac{3}{8}$

(4) $\frac{2}{7}$ × π + $\frac{5}{7}$ × π = ($\frac{2}{7}$ + $\frac{5}{7}$) × π

Record your results.

* * *

On your work sheet insert the operation signs, '×' or '+', in the following exercises to make instances of the dpma. Then, mark the principal operator of the expression that is the left side of the instance with an '↑', and do the same for the right side.

(1) 27 45 33 45 = (27 33) 45

(2) (10 + 17) 45 33 45 = (10 + 17 33) 45

(3) (85 15) (15 × 5 15) = (85 15 × 5) 15

(4) (20 20) (20 20) = (20 20) 20

The pattern for the LEFT distributive principle for multiplication over addition is introduced and a proof of some of its instances given. A method of proving, which foreshadows proofs in Unit II, is given also. Finally, we show that if we had started with the left distributive principle then we could have shown the right distributive principle to be one of its consequences.

[Part 110] [Page 70]

Check your answers.

(1) $25 \times 53 + 53 \times 47 = (25 + 47) \times 53$

(2) $25 \times 53 + 53 \times 47 = 53 \times (25 + 47)$

(3) $25 \times 53 + 25 \times 47 = 25 \times (53 + 47)$

Record your results.

* * *

Look, again, at statement (3) above. As you can see, it doesn't fit the dpma pattern-sentence.

(3) $25 \times 53 + 25 \times 47 = 25 \times (53 + 47)$

[dpma] ___×..... + ~~~×..... = (___+~~~)×.....

However, you probably suspected that even though (3) is not an instance of the dpma, it is a consequence of it and some other principle.

On your work sheet fill the blanks in the following proof that

(3) $25 \times 53 + 25 \times 47 = 25 \times (53 + 47)$.

$25 \times 53 + 25 \times 47 = 53 \times$? $+ 47 \times$? [? , twice]

____?____ = (? + ?) × 25 [?]

$(53 + 47) \times 25$ = __? × ?__ [cpm]

Therefore,

$25 \times 53 + 25 \times 47 =$ ____?____.

So, statement (3) is a consequence of the _?_ and the _?_.

[Part 110] [Page 71]

Check your answers.

Here is a proof that

(3) $25 \times 53 + 25 \times 47 = 25 \times (53 + 47)$.

$25 \times 53 + 25 \times 47 = 53 \times 25 + 47 \times 25$	[cpm]
$53 \times 25 + 47 \times 25 = (53 + 47) \times 25$	[dpma]
$(53 + 47) \times 25 = 25 \times (53 + 47)$	[cpm]

Therefore,

$25 \times 53 + 25 \times 47 = 25 \times (53 + 47)$.

So, statement (3) is a consequence of the cpm and the dpma.

Record your results.

* * *

Here is another statement which has the same pattern as that of statement (3), above.

$999 \times 77 + 999 \times 88 = 999 \times (77 + 88)$

The statement is a consequence of the cpm and the dpma. Write a proof of the statement on your work sheet, following the pattern of the proof in the box, above.

[Part 110] [Page 72]

Check your answers.

Here is a proof that

$999 \times 77 + 999 \times 88 = 999 \times (77 + 88)$.

$999 \times 77 + 999 \times 88 = 77 \times 999 + 88 \times 999$	[cpm]
$77 \times 999 + 88 \times 999 = (77 + 88) \times 999$	[dpma]
$(77 + 88) \times 999 = 999 \times (77 + 88)$	[cpm]

Therefore,

$999 \times 77 + 999 \times 88 = 999 \times (77 + 88)$.

Record your results.

* * *

We have just proved that

(A) $999 \times 77 + 999 \times 88 = 999 \times (77 + 88)$,

and we didn't have to do any computing in proving statement (A). Suppose that someone is willing to accept the proof of statement (A) but then wonders about the statement:

(B) $999 \times 77 + 999 \times 39 = 999 \times (77 + 39)$

Of course, he could just simplify both sides of statement (B). If he did his computing correctly, he would discover that statement (B) is true. But, as you know, there is a much easier way. Just prove statement (B).

On your work sheet is a proof of statement (A). Make some changes in it [Cross-outs and write-ins] so that it becomes a proof of statement (B).

[Part 110] [Page 73]

Check your answer.

Here is a proof that

(B) $999 \times 77 + 999 \times \overset{39}{\cancel{22}} = 999 \times (77 + \overset{39}{\cancel{22}})$.

$999 \times 77 + 999 \times \overset{39}{\cancel{22}} = 77 \times 999 + \overset{39}{\cancel{22}} \times 999$ [cpm]

$77 \times 999 + \overset{39}{\cancel{22}} \times 999 = (77 + \overset{39}{\cancel{22}}) \times 999$ [dpma]

$(77 + \overset{39}{\cancel{22}}) \times 999 = 999 \times (77 + \overset{39}{\cancel{22}})$ [cpm]

Therefore,

$999 \times 77 + 999 \times \overset{39}{\cancel{22}} = 999 \times (77 + \overset{39}{\cancel{22}})$.

You are correct if you made all the changes shown above. Record your result.

* * *

Here is another statement which follows the pattern of statement (B):

(C) $287 \times 65 + 287 \times 43 = 287 \times (65 + 43)$

To get a proof of statement (C), just take the proof of statement (B), and

(1) cross out all the '999's and write '_?_'s in their places;

(2) cross out all the '_?_'s and write '65's in their places; and

(3) cross out all the '39's and write '_?_'s in their places.

When you have finished, you will have a proof that shows that statement (C) is a consequence of the _?_ and the _?_.

[Fill the indicated blanks on your work sheet.]

[Part 110] [Page 74]

Check your answers.

Below is a proof that

(C) $287 \times 65 + 287 \times 43 = 387 \times (65 + 43)$

which follows from of (B) when you

(1) cross out all the '999's and write '287's in their places;

(2) cross out all the '77's and write '65's in their places; and

(3) cross out all the '39's and write '43's in their places.

Here is the proof.

$\overset{287}{\cancel{999}} \times \overset{65}{\cancel{77}} + \overset{287}{\cancel{999}} \times \overset{43}{\cancel{22}} = \overset{65}{\cancel{77}} \times \overset{287}{\cancel{999}} + \overset{43}{\cancel{22}} \times \overset{287}{\cancel{999}}$ [cpm]

$\overset{65}{\cancel{77}} \times \overset{287}{\cancel{999}} + \overset{43}{\cancel{22}} \times \overset{287}{\cancel{999}} = (\overset{65}{\cancel{77}} + \overset{43}{\cancel{22}}) \times \overset{287}{\cancel{999}}$ [dpma]

$(\overset{65}{\cancel{77}} + \overset{43}{\cancel{22}}) \times \overset{287}{\cancel{999}} = \overset{287}{\cancel{999}} \times (\overset{65}{\cancel{77}} + \overset{43}{\cancel{22}})$ [cpm]

Therefore,

$\overset{287}{\cancel{999}} \times \overset{65}{\cancel{77}} + \overset{287}{\cancel{999}} \times \overset{43}{\cancel{22}} = \overset{287}{\cancel{999}} \times (\overset{65}{\cancel{77}} + \overset{43}{\cancel{22}})$.

So, statement (C) is a consequence of the cpm and the dpma

Record your results.

* * *

The exercises you have just done give you a technique for proving a statement which has the pattern shown by:

..... × ___ + ... × ___ = _____?_____

Complete the pattern-sentence on your work sheet.

[Part 110] [Page 77]

Check your answers.

Any instance of the ℓdpma is a consequence of the cpm and the dpma.

Any instance of the dpma is a consequence of the cpm and the ℓdpma

Record your results.

* * *

Suppose that we had discussed the ℓdpma before discussing the dpma. Then, here is how we would have proved the statement:

(☆) $33 \times 55 + 44 \times 55 = (33 + 44) \times 55$

which is an instance of the dpma. [Fill the blanks on your work sheet.]

$33 \times 55 + 44 \times 55 =$? $\times 33 +$? $\times 44$ [?]

_____?_____ = ? $\times (33 + 44)$ [?]

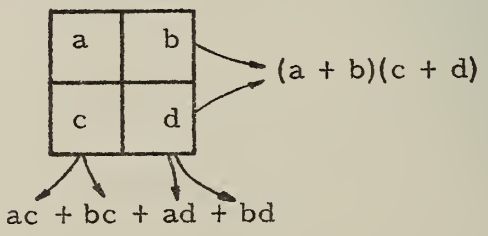
_____?_____ = $(33 + 44) \times$? [?]

Therefore,

(☆) $33 \times 55 + 44 \times 55 =$ _____?_____.

So, statement (☆) is a consequence of the _?_ and the _?_.

A rather simple chart which leads into a rather interesting and complex proof is given. The pattern is this:



[Part 110] [Page 82]

As you might suspect the fact that the two ways of working with the 2-by-2 chart lead to the same result is not a lucky coincidence. Actually it is a consequence of principles you have learned about in this part.

Let's look at a 2-by-2 chart with complicated numbers.

99	77
66	44

First way of working:

99	77
66	44

99	77
66	44

gives this result:

$[(99 \times 66) + (77 \times 66)] + [(99 \times 44) + (77 \times 44)]$

Second way of working:

99	77
66	44

99	77
66	44

gives this result: $(99 + 77) \times (66 + 44)$

Conclusion:

$[(99 \times 66) + (77 \times 66)] + [(99 \times 44) + (77 \times 44)] = (\ ?\ ) \times (\ ?\ )$

[Fill the blanks on your work sheet.]

[Part 110] [Page 83]

Check your answers and record your results.

$[(99 \times 66) + (77 \times 66)] + [(99 \times 44) + (77 \times 44)] = (99 + 77) \times (66 + 44)$

Now, on your work sheet, fill the blanks in the following proof of the above statement.

$[(99 \times 66) + (77 \times 66)] + [(99 \times 44) + (77 \times 44)]$

$= [(99 + \ ?\ ) \times 66] + [(99 + 77) \times 44]$ [?]

$[(\ 7 + ?\ ) \times 66] + [(\ ? + ?\ ) \times 44]$

$= (99 + 77) \times (\ ? + ?\ )$ [?]

Therefore,

$[(99 \times 66) + (\ 7 \times ?\ )] + [(\ ? \times ?) + (77 \times 44)]$

$= (\ ? + ?\ ) \times (66 + 44)$

The statement to be proved is a consequence of the __?__ and the __?__.

[Part 110] [Page 84]

Check your answers.

Here again, is the proof that

$[(99 \times 66) + (77 \times 66)] + [(99 \times 44) + (77 \times 44)] = (99 + 77) \times (66 + 44)$.

$[(99 \times 66) + (77 \times 66)] + [(99 \times 44) + (77 \times 44)]$

$= [(99 + 77) \times 66] + [(99 + 77) \times 44]$ [dpma]

$[(99 + 77) \times 66] + [(99 + 77) \times 44]$

$= (99 + 77) \times (66 + 44)$ [ldpma]

Therefore,

$[(99 \times 66) + (77 \times 66)] + [(99 \times 44) + (77 \times 44)] = (99 + 77) \times (66 \times 44)$,

which is a consequence of the dpma and the ldpma.

Record your results.

* * *

We may use the conclusion, above, to simplify:

$[(99 \times 66) + (77 \times 66)] + [(99 \times 44) + (77 \times 44)]$

Instead, we just simplify:

$\underbrace{(99 + 77)}_{176} \times \underbrace{(66 + 44)}_{110}$

and get: $\underbrace{176 \quad 110}_{19360}$

$$\begin{array}{r} 176 \\ \times 110 \\ \hline 1760 \\ 176 \\ \hline 19360 \end{array}$$

Now, you simplify:

$[(37 \times 75) + (52 \times 75)] + [(37 \times 25) + (52 \times 25)]$

Write the answer on your work sheet.

PART 110.5

The proofs in the preceding parts have been limited to relatively simple proofs. Now, we wish to develop the students' skill in handling more complex proofs. There is considerable emphasis on the fact that a proof shows that its conclusion is a consequence of certain mathematical principles. For this reason, we include a proof of a false statement based on an assumption of a false statement, explaining that the PROOF is nevertheless valid.

[Part 110.5] [Page 6]

Check your answers and record your results.

Here are the statements about numbers of arithmetic:

$87 \times 4899 \times 79 = 79 \times 87 \times 4899$

$79 \times 87 \times 4899 = 6783 \times 4899$

$6783 \times 4899 = 33,229,917$

(1) [No] The statement:
$87 \times 4899 \times 79 = 79 \times 4899 \times 87$
is not a logical consequence of just these statements shown above.

(2) [No] The statement:
$79 \times 4899 \times 87 = 33,229,917$
is not a logical consequence of these statements, either.

(3) [Yes] The statement:
(☆) $87 \times 4899 \times 79 = 6783 \times 4899$
is a logical consequence of the first two given statements. That is:
$87 \times 7899 \times 79$ is the same as $79 \times 87 \times 4899$,
which is the same as 6783×4899.

* * *

Let us reexamine the three statements at the top of this page. Mark each one to show whether you think it is 'True', 'False', or '?' [if you are not sure whether it is a true or a false statement]

(1) $87 \times 4899 \times 79 = 79 \times 87 \times 4899$

(2) $79 \times 87 \times 4899 = 6783 \times 4899$

(3) $6783 \times 4899 = 33,229,917$

Next, fill the blanks in the following sentence on your work sheet:
The statement from Exercise 3, above:
(☆) $87 \times 4899 \times 79 = 6783 \times 4899$
__?__ a logical consequence of the statements (1), (2), and (3).
[is/is not]

[Part 110.5] [Page 7]

Check your answers.

We asked you to mark what you THOUGHT about each of the statements:

(1) $87 \times 4899 \times 79 = 79 \times 87 \times 4899$

(2) $79 \times 87 \times 4899 = 6783 \times 4899$

(3) $6783 \times 4899 = 33,229,917$

You are correct if you answered what you THOUGHT. But, regardless of whether you thought statements (1), (2), and (3) were true or false, the statement:

(☆) $87 \times 4899 \times 79 = 6783 \times 4899$

is a logical consequence of the statements (1) and (2), above.

Record your results.

* * *

You may be surprised at what we discovered about the statements above.

(1) $87 \times 4899 \times 79 = 79 \times 87 \times 4899$ [True]

(2) $79 \times 87 \times 4899 = 6783 \times 4899$ [False]

(3) $6783 \times 4899 = 33,229,917$ [True]

Here is why statement (2) is false.

$$\begin{array}{r} 79 \\ \times 87 \\ \hline 553 \\ 632 \\ \hline 6873 \end{array}$$

[not '6783' as shown in (2): $79 \times 87 \times 4899 = 6783 \times 4899$]

Does this discovery now mean that it is FALSE to say that the statement:

(☆) $87 \times 4899 \times 79 = 6783 \times 4899$

is a logical consequence of statements (1) and (2), above? [Yes/No]

Another point to be emphasized is that if a proof is valid then the conclusion must be accepted if the principles on which it is based are accepted.

[Part 110.5] [Page 24]

Check your proof.

Here is a proof of the statement:

$[(7 + 6) \times 7] + (6 \times 6) = (7 \times 7) + [6 \times (7 + 6)]$

$[(7 + 6) \times 7] + (6 \times 6) = [(7 \times 7) + (6 \times 7)] + (6 \times 6)$ [dpma]

$= (7 \times 7) + [(6 \times 7) + (6 \times 6)]$ [apa]

$= (7 \times 7) + [6 \times (7 + 6)]$ [ldpma]

Therefore,

$[(7 + 6) \times 7] + (6 \times 6) = (7 \times 7) + [6 \times (7 + 6)]$.

Record your result.

* * *

(1) Give a proof of the statement:

$(792 + 792) \times 584 = 792 \times (584 + 584)$

(2) After you prove the statement, tell whether it is true or false.

[Part 110.5] [Page.25]

Check your answers.

(1) Here is a proof of the statement:

$(792 + 792) \times 584 = 792 \times (584 + 584)$

$(792 + 792) \times 584 = (792 \times 584) + (792 \times 584)$ [dpma]

$= 792 \times (584 + 584)$ [ldpma]

Therefore,

$(792 + 792) \times 584 = 792 \times (584 + 584)$

(2) To prove a statement means to show that it is a logical consequence of certain principles. In the proof above, you see that the statement is a logical consequence of the dpma and the ldpma. If you believe that those principles are true then, logically, you must believe that the given statement is *true* .

Record your results.

* * *

Answer these questions on your work sheet.

(1) $21 \times 9 + 21 = ?$

(2) $4 \times 9 + 4 = ?$

(3) $6 \times 9 + 6 = ?$

(4) $7 \times 9 + 7 = ?$

(5) $1776 \times 9 + 1776 = ?$

Numerous exercises are provided in which the student learns to recognize instances of the principles for multiplying by one, by zero, and for adding zero. Then they are asked to discriminate between them in their own proof of a statement.

[Part 110.5] [Page 44]

Check your answer.

(1) '1 + 0 = 1' is an instance of the principle for adding zero. It tells us that adding 0 to 1 gives 1.

(2) '0 + 1 = 1' is not an instance of the pa 0. Instead, it merely tells us that adding 1 to 0 gives 1..

Record your result.

* * *

Some of the following statements are instances of principles that we have discussed, and some are not. For those statements that are instances, tell which principles. Leave the others blank.

(1) $3 \times 1 = 3$ [?]

(2) $1 \times 3 = 3$ [?]

(3) $6 \times 0 = 0$ [?]

(4) $6 + 0 = 6$ [?]

(5) $(3 \times 0) \times 1 = 3 \times 0$ [?]

(6) $1 \times 0 = 0$ [?]

[Part 110.5] [Page 45]

Check your answers.

(1) $3 \times 1 = 3$ [pm 1]

(2) $1 \times 3 = 3$ [not an instance of one of our principles]

(3) $6 \times 0 = 0$ [pm 0]

(4) $6 + 0 = 6$ [pa 0]

(5) $(3 + 0) \times 1 = 3 + 0$ [pm 1]

(6) $1 \times 0 = 0$ [pm 0]

Record your results.

* * *

Prove the statement:

$$(3 \times 1) + 0 = 3$$

by using the fact that it is a logical consequence of the pa 0 and the pm 1.

[Part 110.5] [Page 46]

Check your proof.

Here is a proof of the statement:

$$(3 \times 1) + 0 = 3$$

$(3 \times 1) + 0 = 3 \times 1$ [pa 0]

$= 3$ [pm 1]

Therefore,

$(3 \times 1) + 0 = 3.$

Record your result.

* * *

Prove the statement:

$$(1 \times 0) + (0 \times 1) = 0$$

[Part 110.5] [Page 47]

Check your proof.

Here is a proof of the statement:

$$(1 \times 0) + (0 \times 1) = 0$$

$(1 \times 0) + (0 \times 1) = (1 \times 0) + 0$ [pm 1]

$= 0 + 0$ [pm 0]

$= 0$ [pa 0]

Therefore,

$(1 \times 0) + (0 \times 1) = 0.$

Here is another proof:

$(1 \times 0) + \underline{(0 \times 1)} = (1 \times 0) + \underline{(1 \times 0)}$ [cpm]

$= (1 + 1) \times 0$ [dpma]

$= 0$ [pm 0]

Therefore,

$(1 \times 0) + (0 \times 1) = 0.$

Your proof is correct if the conclusion is the same as ours and if it is a logical consequence of <u>just</u> the principle that we have discussed.

Record your result.

* * *

Prove the statement:

$$[(75 \times 1) \times 99] + [75 + (25 \times 0)] = 100 \times 75$$

We have now presented all the principles for numbers of arithmetic from Unit I without mentioning principles for REAL numbers. After a set of review exercises, we make the transition to the principles for real numbers. We attempt to capitalize on the previously-learned pattern-sentences, and establish the same level of confidence in the principles for real numbers as existed for principles of numbers of arithmetic.

[Part 110.5] [Page 50]

Check your answers.

(1) $^{+}8 + {}^{-}5 = {}^{+}3$

(2) ${}^{-}7 + 4 = {}^{-}11$

(3) ${}^{-}5 + {}^{+}8 = {}^{+}3$

(4) $4 + {}^{-}7 = {}^{-}11$

(5) ${}^{+}7 + 18 = {}^{-}11$

(6) $(74 + {}^{-}7) + 18 = {}^{-}85$

(7) ${}^{-}86 + {}^{+}99 = {}^{+}13$

(8) $({}^{+}100 + 86) + {}^{+}99 = {}^{+}113$

Record your results.

* * *

Now, do these problems. Write the answers on your work sheet.

(1) ${}^{+}38 + 75 = ?$

(2) $37 + 25 = ?$

(3) ${}^{-}75 + 25 = ?$

(4) ${}^{+}38 + {}^{-}100 = ?$

(5) $({}^{+}38 + 75) + 25 = ?$

[Part 110.5] [Page 51]

Check your answers.

(1) ${}^{+}38 + {}^{-}75 = {}^{-}37$

(2) ${}^{-}37 + 25 = {}^{-}62$

(3) ${}^{-}75 + {}^{-}25 = {}^{-}100$

(4) ${}^{+}38 + {}^{-}100 = {}^{-}62$

(5) $({}^{+}38 + {}^{-}75) + {}^{-}25 = {}^{-}62$

Record your results.

* * *

The fact that Exercises (2) and (4), above, have the same answer tells us that

$$({}^{+}38 + 75) + {}^{-}25 = {}^{+}38 + (75 + 25).$$

This gives us some evidence that addition of real numbers is __?__.
[Fill the blank on your work sheet.]

Check your answer.

The pattern-sentence for the associative principle for addition of real numbers is:

$$(__ + ___) + \ldots\ldots = __ + (___ + \ldots\ldots)$$

Record your result.

* * *

Use the associative principle for addition of real numbers to simplify the following:

(1) $(^{-}75 + {}^{+}9) + {}^{+}91$

(2) $(^{+}50 + {}^{-}33) + {}^{-}67$

(3) $(^{+}31 + {}^{-}34) + {}^{+}134$

(4) $(^{-}287 + {}^{-}25) + {}^{+}24$

(5) $^{-}28 + (^{+}28 + {}^{-}75)$

(6) $^{-}25 + (^{-}50 + {}^{+}75)$

(7) $^{+}397 + (^{+}189 + {}^{-}397)$

(8) $(^{-}237 + {}^{+}626) + (^{-}626 + {}^{+}237)$

Check your answers.

3	7	⇒ 10
5	2	⇒ 7

(1) ROW TOTAL: 17

3	7
5	2

(2) COLUMN TOTAL: 17

Record your results.

* * *

You should not be surprized that the ROW TOTAL is equal to the COLUMN TOTAL in the exercise above. We proved they were equal in a similar problem earlier, by showing that this fact was a logical consequence of the apa and the cpa for numbers of arithmetic.

Here is a 2-by-2 chart that uses real numbers.

$^{-}13$	$^{+}8$
$^{+}9$	5

(1) Do you think that the ROW TOTAL is, or, is not, equal to the ROW COLUMN?

After you have answered Exercise (1), then do the next two problems.

(2) Compute the ROW TOTAL.

(3) Compute the COLUMN TOTAL.

Check your answers.

$^{-}13$	$^{+}8$	⇒ $(^{-}13 + {}^{+}8)$
$^{+}9$	$^{-}5$	⇒ $(^{+}9 + {}^{-}5)$
⇓ $(^{-}13 + {}^{+}9)$	⇓ $(^{+}8 + {}^{-}5)$	

(1) Using the apa and the cpa for real numbers, we can prove that the ROW TOTAL is equal to the COLUMN TOTAL for the 2-by-2 chart above.

ROW TOTAL: $(^{-}13 + {}^{+}8) + (^{+}9 + {}^{-}5)$

COLUMN TOTAL: $(^{-}13 + {}^{+}9) + (^{+}8 + {}^{-}5)$

Here is a proof:

$$\begin{aligned}
(^{-}13 + {}^{+}8) + (^{+}9 + {}^{-}5) &= [(^{-}13 + {}^{+}8) + {}^{+}9] + {}^{-}5 && \text{[apa]}\\
&= [\ 13 + (^{+}8 + {}^{+}9)] + {}^{-}5 && \text{[apa]}\\
&= [\ 13 + (^{+}9 + {}^{+}8)] + {}^{-}5 && \text{[cpa]}\\
&= [(\ 13 + {}^{+}9) + {}^{+}8] + {}^{-}5 && \text{[apa]}\\
&= (^{-}13 + {}^{+}9) + (^{+}8 + {}^{-}5) && \text{[apa]}
\end{aligned}$$

Therefore,

$$\underbrace{(\ 13 + {}^{+}8) + (^{+}9 + {}^{-}5)}_{\text{ROW TOTAL}} = \underbrace{(^{-}13 + {}^{+}9) + (^{+}8 + {}^{-}5)}_{\text{COLUMN TOTAL}}.$$

Of course, you probably did not use such a formal proof to answer this question. Instead, we hoped that you would merely realize that the desired conclusion was a logical consequence of the apa and the cpa for real numbers. [Do not record the result for this problem.]

(2) ROW TOTAL:

(3) COLUMN TOTAL:

$^{-}13$	$^{+}8$	⇒ $^{-}5$
$^{+}9$	5	⇒ $^{+}4$
⇓ $^{-}4$	⇓ $^{+}3$ ⇒	$^{-}1$

Record your results for Exercises (2) and (3).

* * *

Do these multiplication problems.

(1) $^{-}4 \times {}^{-}25 = ?$

(2) $^{-}100 \times {}^{-}18 = ?$

(3) $^{-}25 \times {}^{-}18 = ?$

(4) $4 \times {}^{-}450 = ?$

(5) $4 \times (^{-}25 \times {}^{-}18) = ?$

Check your answers.

$^{-}3 \times {}^{-}7 = 7 \times {}^{-}3$

$^{+}8 \times {}^{-}9 = {}^{-}9 \times {}^{+}8$

$^{-}14 \times (^{+}12 \times {}^{-}17) = {}^{-}14 \times (^{-}17 \times {}^{+}12)$

$(^{+}7 + {}^{-}4) \times (^{-}3 + {}^{-}5) = (^{-}3 + 5) \times (^{+}7 + {}^{-}4)$

$(^{-}4 \times {}^{+}3) + (^{+}2 \times {}^{-}1) = (^{+}3 \times {}^{-}4) + (^{-}1 \times {}^{+}2)$

Each of the above statements is a logical consequence of the commutative principle for multiplication of real numbers.

Record your results.

* * *

You should now be aware of the fact that

addition of real numbers is commutative and associative,

and that

multiplication of real numbers is commutative and associative.

Here is a pattern-sentence for the distributive principle for multiplication over addition for numbers of arithmetic:

$$(__ \times \ldots\ldots) + (__ \times \ldots\ldots) = (__ + __) \times \ldots\ldots$$

Answer this question:

Does multiplication of real numbers distribute over addition of real numbers?

The following exerpts are taken from the tests for parts 111, 112, 113, 114, 114.5, 115 and 116. These parts are presented in the next section.

* * *

True or false.

T F Joe is thinking of an operation that has an inverse. The inverse of the inverse of Joe's operation is <u>certain</u> to be an operation.

T F The opposite of the absolute value of a number is the same as the absolute value of the opposite of the number.

- -

Multiple-choice.

A B C D The problem:

$^{+}11 - {}^{+}5$

involves a certain operation. The inverse of that operation is

A) subtracting $^{+}11$
B) adding the opposite of $^{+}5$
C) adding $^{+}5$
D) subtracting the opposite of $^{-}5$

A B C D If a first real number is the opposite of a second real number, then

A) their difference is nonnegative
B) their quotient is 1
C) their product is nonpositive
D) none of these

- -

The '⌽', '⌂', and other numerals used in the sentences shown below are from the planet Glox. We know that each numeral names a number, but we don't know <u>which</u> number corresponds to any particular numeral. (The signs '>', '≠', etc., have their usual earth meanings.)

A B C If '▽ < ⌽' is true, '⌽ ≥ ⌂' is

A) true
B) false
C) cannot tell

A B C If '⊕ < ▽' is true, '⊕ ≮ ▽' is

A) true
B) false
C) cannot tell

[Test excerpts continued.]

Simplify.

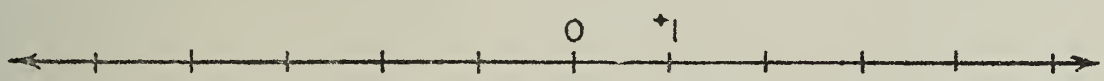
$$({}^{-}6.3 + {}^{+}1.4) + - ({}^{+}1.4 + {}^{-}6.3) + {}^{-}3 \times -\frac{{}^{-}1}{3} - {}^{+}1$$

$$|{}^{-}134| + |{}^{+}134|$$

$$|\ ?\ | = {}^{-}5$$

The reciprocal of a certain number is the same as the opposite of the opposite of the number. The number is __?__ .

A picture of the number line is given below. Put an 'x' on the mark which corresponds to $^{-}3$, and make at least four heavy dots to mark points which are $< {}^{-}3$.

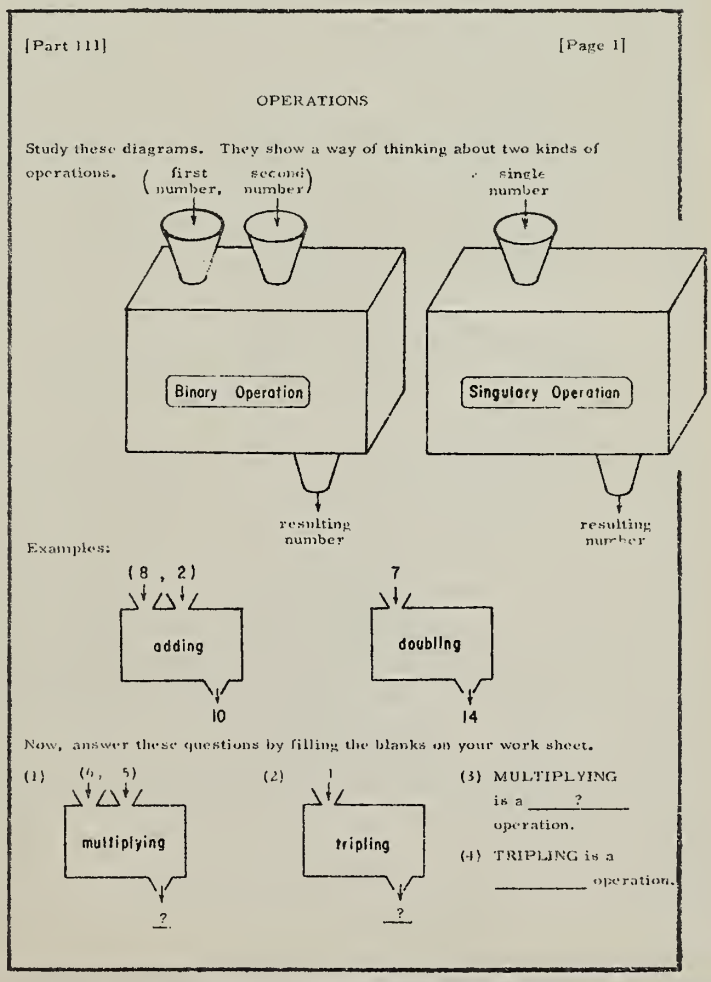


PART 111

Machine notation serves two functions in the opening pages of Part 111.

On pages 1-3, it is used to fortify the distinction between a binary and a singulary operation. The absence of discussion on those pages is testimony to the usefulness of the notation.

On pages 5-8, machines are used in a step-by-step development of list notation.


[Part 111] [Page 1]

OPERATIONS

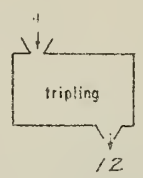
Study these diagrams. They show a way of thinking about two kinds of operations.

[Part 111] [Page 2]

Check your answers.

(1)

(2)



(3) For each PAIR OF NUMBERS to which multiplication is applied, you get a unique [exactly one] result. This is why we say that multiplying is a *binary* operation.

(4) For each SINGLE NUMBER to which tripling is applied, you get a unique result. This is why we say that tripling is a *singulary* operation.

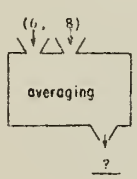
Record your results.

* * *

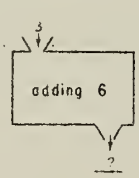
[Pronounce 'singulary' as 'SING-you-lerry'.]

Answer these questions on your work sheet.

(1)



(2)



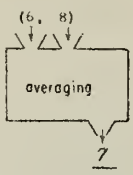
(3) AVERAGING is a _____ operation.

(4) ADDING 6 is a _____ operation.

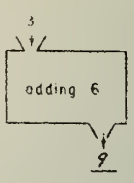
[Part 111] [Page 3]

Check your answers.

(1)



(2)



(3) Each PAIR of numbers has a unique average. In other words, averaging is a *binary* operation.

(4) For each SINGLE NUMBER to which you add 6, you get a unique result. In other words, adding 6 is a *singulary* operation.

Record your results.

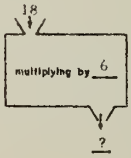
* * *

On your work sheet, write the answers to these questions.

(1)

(2)

(3)



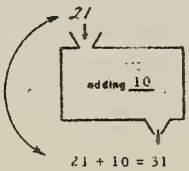
(4)

[Part 111] [Page 5]

Check your answers.

(1) 9 + 10 = 19

(2)



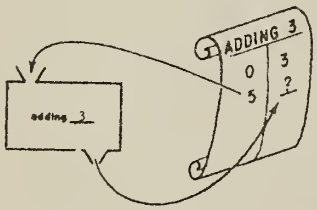
21 + 10 = 31

In earlier parts, you learned about BINARY OPERATIONS such as addition and multiplication. Binary operations take you from pairs of numbers to single numbers. [For example, the binary operation addition takes you from (8, 4) to 12.]

A singulary operation takes you from a single number to a single number. [Adding 10 takes you from 9 to 19.]

Record your results.

* * *

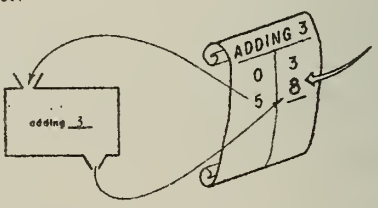


Look at this list. It is another way of thinking about singulary operations.

Fill the blank on your work sheet.

[Part 111] [Page 6]

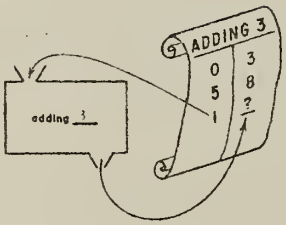
Check your answer.



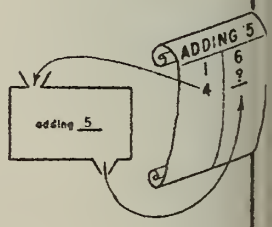
Record your result.

* * *

(1)



(2)



Fill the blanks on the scrolls pictured on your work sheet.

[Part III] [Page 7]

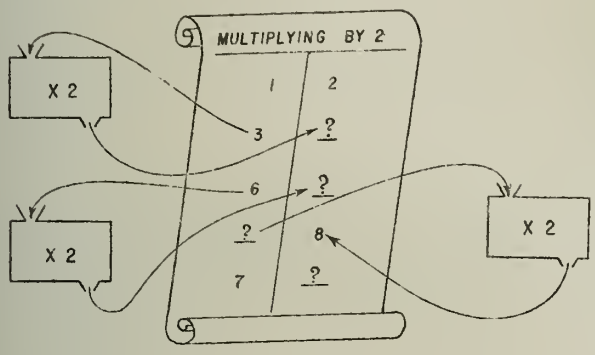
Check your answers.

(1) 1 + 3 = 4 (2) 4 + 5 = 9

Record your results.

* * *

Here is another singulary operation to consider.



Complete the list on your worksheet.

[Part III] [Page 8]

Here is another list for the singulary operation multiplying by 2. It contains the same pairs of numbers as the list on page 7a, and in each case the FIRST number of the pair is the one from the left-hand column of the list on page 7a.

multiplying by 2

(1, 2)
(3, 6)
(6, 12)
(4, 8)
(7, 14)
. . .

[The three dots at the bottom of the list indicate that there are more pairs which belong in the list.]

Even someone who didn't know how to find the answers to multiplication problems could use this list to do the problem:

4 X 2 = ?

The list tells you that the answer is: 8

4 is the FIRST NUMBER of a pair whose SECOND NUMBER is 8

Answer these questions.

(1) Using the list, we can see that 7 × 2 is 14 since 7 is the ___?___ number of a pair whose second number is ___?___.

(2) Notice that the pair (3, 6) belongs to the list.
3 is the first number of the pair and
6 is the ___?___ number.
From this, you can conclude that
___?___ is the product of ___?___ and 2.

The list notation just developed makes it possible to discuss an operation by discussing lists of ordered pairs which belong to that operation. In Part III we propose to have the students examine lists of pairs belonging to singulary operations. Suppose, then, that * and ☆ are singulary operations on a set S such that applying ☆ to an element of S undoes what applying * to that element did. We say that:

☆ is the inverse of *

By having the student think of ☆ and * as sets of ordered pairs, we hope he will discover that the pairs in ☆ are "the pairs in * turned around." To accomplish this, we must call the student's attention to the ordered pairs in ☆ and *. That is, we must give the student some reason to use the lists. Since there are few students who would actually use the list for MULTIPLYING BY 3 to solve '6 × 3 = ?', we define an operation, ZIFFING, for which the relationship between the components of its ordered pairs is not so obvious.

[Part III] [Page 21]

Here is a list for another singulary operation. Let's call this singulary operation ZIFFING. [Just a word we made up!]

ZIFFING

(0,0) $(\frac{3}{2}, \frac{15}{4})$ (2,6) $(\frac{1}{4}, \frac{5}{16})$ $(\frac{1}{2}, \frac{3}{4})$ (1,2)

$(\frac{2}{3}, \frac{10}{9})$ (81, 6642) (0.1, 0.11) (11, 132) (7,56)

(6, 42) $(\frac{3}{4}, \frac{21}{16})$ $(2\frac{1}{2}, 8\frac{3}{4})$ (0.9, 1.71) (16, 272)

(0.2, 0.24) (42, 1806) (3, 12) $(0.6, \frac{88}{100})$

(5, 30) . . .

Let's use the letter 'z' to refer to this singulary operation. For instance, instead of saying that the ziff of 2 is 11, we could say:

z(2) = 6 [read as 'zee of 2 is 6']

As with other lists for singulary operations, we can use this one to do some problems in ziffing. For example, we can conclude that z(16) = 272 because (16, 272) is contained in the ziffing list.

Use the ziffing list to answer these questions.

(1) The ziff of 1 is ___?___.

(2) $z(\frac{2}{3})$ = ___?___

(3) z(6) = ___?___

(4) z(___?___) = 6

[Part III] [Page 22]

Check your answers.

ZIFFING

(0,0) $(\frac{3}{2}, \frac{15}{4})$ (2,6) $(\frac{1}{4}, \frac{5}{16})$ $(\frac{1}{2}, \frac{3}{4})$ (1,2)

$(\frac{2}{3}, \frac{10}{9})$ (81, 6642) (0.1, 0.11) (11, 132) (7,56)

(6, 42) $(\frac{3}{4}, \frac{21}{16})$ $(2\frac{1}{2}, 8\frac{3}{4})$ (0.9, 1.71) (16, 272)

(0.2, 0.24) (42, 1806) (3, 12) $(0.6, \frac{88}{100})$

(5, 30) . . .

(1) z(1) = ___2___

(2) The ziff of $\frac{2}{3}$ is ___10/9___.

(3) z(6) = ___42___

(4) z(___2___) = 6

Record your results.

* * *

Use the ziffing list to answer these questions.

(1) z(1) + z(2) = ___?___

(2) z(1) + z(7) = ___?___

(3) z(3) + z(5) = z(___?___)

(4) z(3) divided by z(2) = z(___?___)

[Part III] [Page 23]

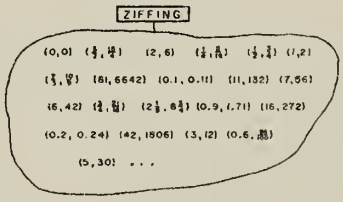
Check your answers.

(1) z(1) + z(2) = ___8___
2 + 6

(2) z(1) + z(7) = ___58___
2 + 56

(3) z(3) + z(5) = z(___6___)
12 + 30 = 42

(4) z(3) divided by z(2) = z(___1___)
12 ÷ 6 = 2



Record your results.

* * *

Here is part of the ziffing list.

(2, 6) (1, 2)

(6, 42) $(\frac{1}{2}, \frac{3}{4})$ $(\frac{3}{4}, \frac{21}{16})$

(42, 1806)

We see from this list that z(2) = 6 [the ziff of 2 is 6]
and z(6) = 42 [the ziff of 6 is 42].
So, z(z(2)) = 42 [the ziff of the ziff of 2 is 42].

Answer the following on your work sheet.

(1) z(1) = ___?___ and z(2) = ___?___. So, z(z(1)) = ___?___.

(2) $z(z(\frac{1}{2}))$ = ___?___

(3) z(z(6)) = ___?___

[Page 23a] [Part III]

Check your answers.

ZIFFING

(0,0) $(\frac{3}{2}, \frac{15}{4})$ (2,6) $(\frac{1}{4}, \frac{5}{16})$ $(\frac{1}{2}, \frac{3}{4})$ (1,2)

$(\frac{2}{3}, \frac{10}{9})$ (81, 6642) (0.1, 0.11) (11, 132) (7,56)

(6, 42) $(\frac{3}{4}, \frac{21}{16})$ $(2\frac{1}{2}, 8\frac{3}{4})$ (0.9, 1.71) (16, 272)

(0.2, 0.24) (42, 1806) (3, 12) $(0.6, \frac{88}{100})$

(5, 30) . . .

(1) z(1) = ___2___ and z(2) = ___6___. So, z(z(1)) = ___6___.

(2) $z(z(\frac{1}{2}))$ = $z(\frac{3}{4})$ = ___21/16___

(3) z(z(6)) = z(42) = ___1806___

Record your results.

* * *

Now read PAGE 24.

[Part III] [Page 24]

Here is a list for another singulary operation. We have called this operation FIZZING.

FIZZING

(0,0) $(\frac{15}{4}, \frac{3}{2})$ (6,2) $(\frac{5}{16}, \frac{1}{4})$ $(\frac{3}{4}, \frac{1}{2})$ (2,1)

$(\frac{10}{9}, \frac{5}{3})$ (6642, 81) (0.11, 0.1) (132, 11) (56, 7)

(42, 6) $(\frac{21}{8}, \frac{3}{4})$ $(8\frac{3}{4}, 2\frac{1}{2})$ (1.71, 0.9) (272, 16)

(0.24, 0.2) (1806, 42) (12, 3) $(\frac{36}{100}, 0.6)$

(30, 5) . . .

We shall use the letter 'f' to refer to fizzing. So, for example, when we say that f(6) = 2, we mean that the fizz of 6 is 2.

Use just the fizzing list, above, to answer the following questions.

(1) The fizz of 6642 is ___?___.

(2) f(1.71) = ___?___

(3) f(f(42)) = ___?___

[Part III] [Page 25]

Check your answers.

(1) f(6642) = 81

(2) The fizz of (1.71) = 0.9.

(3) f(f(42)) = f(6) = 2

Record your results.

* * *

ZIFFING

(0,0) $(\frac{3}{2}, \frac{15}{4})$ (2,6) $(\frac{1}{4}, \frac{5}{16})$ $(\frac{1}{2}, \frac{3}{4})$ (1,2)

$(\frac{5}{3}, \frac{10}{9})$ (81, 6642) (0.1, 0.11) (11, 132) (7, 56)

(6, 42) $(\frac{3}{4}, \frac{21}{8})$ $(2\frac{1}{2}, 8\frac{3}{4})$ (0.9, 1.71) (16, 272)

(0.2, 0.24) (42, 1806) (3, 12) $(0.6, \frac{36}{100})$

(5, 30) . . .

FIZZING

(0,0) $(\frac{15}{4}, \frac{3}{2})$ (6,2) $(\frac{5}{16}, \frac{1}{4})$ $(\frac{3}{4}, \frac{1}{2})$ (2,1)

$(\frac{10}{9}, \frac{5}{3})$ (6642, 81) (0.11, 0.1) (132, 11) (56, 7)

(42, 6) $(\frac{21}{8}, \frac{3}{4})$ $(8\frac{3}{4}, 2\frac{1}{2})$ (1.71, 0.9) (272, 16)

(0.24, 0.2) (1806, 42) (12, 3) $(\frac{36}{100}, 0.6)$

(30, 5) . . .

Use the lists to answer the following questions.

(1) f(56) + z(2) = ___?___

(2) z(16) + f(0.24) = ___?___

(3) f(6) + z(2) = ___?___

(4) z(6) + f(42) = ___?___

(5) f(z(2)) = ___?___

(6) z(f(42)) = ___?___

(7) $z(f(8\frac{3}{4}))$ = ___?___

[Part III] [Page 27]

Check your answers.

(1) f(56) = 7

(2) z(5) = 30

(3) z(7) = 56

(4) f(30) = 5

Record your results.

* * *

ZIFFING

(0,0) $(\frac{3}{2}, \frac{15}{4})$ (2,6) $(\frac{1}{4}, \frac{5}{16})$ $(\frac{1}{2}, \frac{3}{4})$ (1,2)

$(\frac{5}{3}, \frac{10}{9})$ (81, 6642) (0.1, 0.11) (11, 132) (7, 56)

(6, 42) $(\frac{3}{4}, \frac{21}{8})$ $(2\frac{1}{2}, 8\frac{3}{4})$ (0.9, 1.71) (16, 272)

(0.2, 0.24) (42, 1806) (3, 12) $(0.6, \frac{36}{100})$

(5, 30) . . .

FIZZING

(0,0) $(\frac{15}{4}, \frac{3}{2})$ (6,2) $(\frac{5}{16}, \frac{1}{4})$ $(\frac{3}{4}, \frac{1}{2})$ (2,1)

$(\frac{10}{9}, \frac{5}{3})$ (6642, 81) (0.11, 0.1) (132, 11) (56, 7)

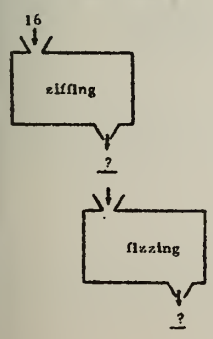
(42, 6) $(\frac{21}{8}, \frac{3}{4})$ $(8\frac{3}{4}, 2\frac{1}{2})$ (1.71, 0.9) (272, 16)

(0.24, 0.2) (1806, 42) (12, 3) $(\frac{36}{100}, 0.6)$

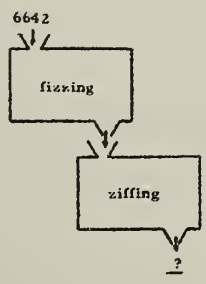
(30, 5) . . .

Answer these questions.

(1)



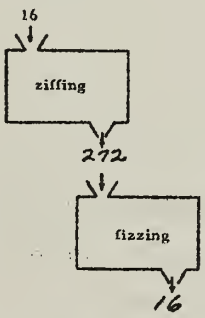
(2)



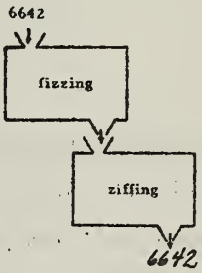
[Part III] [Page 28]

Check your answers.

(1)



(2)



Record your results.

* * *

The problems on the last few pages illustrate an interesting relationship between fizzing and ziffing.

Fizzing undoes what ziffing does.
Ziffing undoes what fizzing does.

By this we mean, for example, that if we ziff 16 to get 272, and then fizz 272, we get 16, the number we started with.

Also, if we fizz 6642, we get some number whose ziff is ___?___.

Fill the blank on your work sheet.

[Part III] [Page 29]

Check your answer.

Ziffing undoes what fizzing does.
Fizzing undoes what ziffing does.

So, if we fizz 6642, we get some number whose ziff is 6642, the number we started with.

Record your result.

* * *

Whenever two operations are related in this manner (that is, whenever one operation undoes what the other operation did), we say that the first operation is the INVERSE of the second.

Thus, since fizzing undoes what ziffing did, fizzing is the INVERSE of ziffing.

(1) Since ziffing ___?___ what ___?___ did, ziffing is the ___?___ of fizzing.

(2) Ziffing is the inverse of fizzing. This means that if the fizz of 12 is 3, then the ziff of 3 is ___?___.

(3) If the ziff of 0.6 is $\frac{96}{100}$, then the fizz of $\frac{96}{100}$ is ___?___.

(4) Assuming that 18 has a ziff, the fizz of the ziff of 18 is ___?___.

(5) Assuming that 18 has a fizz, the ziff of the fizz of 18 is ___?___.

[Part III] [Page 31]

Let's look at the fizzing and ziffing lists again.

ZIFFING

(0,0) $(\frac{3}{2}, \frac{99}{4})$ (2,6) $(\frac{1}{4}, \frac{3}{16})$ $(\frac{1}{2}, \frac{3}{4})$ (1,2)
$(\frac{5}{3}, \frac{27}{9})$ (81,6642) (0.1, 0.11) (11,132) (7,56)
(6,42) $(\frac{3}{4}, \frac{15}{16})$ $(2\frac{1}{2}, 8\frac{1}{4})$ (0.9, 1.71) (16,272)
(0.2, 0.24) (42,1806) (3,12) (0.6, $\frac{96}{100}$)
(5,30) . . .

FIZZING

(0,0) $(\frac{99}{4}, \frac{3}{2})$ (6,2) $(\frac{3}{16}, \frac{1}{4})$ $(\frac{3}{4}, \frac{1}{2})$ (2,1)
$(\frac{99}{9}, \frac{5}{3})$ (6642, 81) (0.11, 0.1) (132, 11) (56, 7)
(42,6) $(\frac{15}{16}, \frac{3}{4})$ $(8\frac{1}{4}, 2\frac{1}{2})$ (1.71, 0.9) (272, 16)
(0.24, 0.2) (1806, 42) (12, 3) ($\frac{96}{100}$, 0.6)
(30, 5) . . .

You have probably already noticed the relationship between these two lists. For example, 2 is the first number of that pair in ziffing whose second number is 6. Also, 6 is the first number of that pair in fizzing whose second number is 2. It is this fact which suggests the name 'inverse' for the relationship between fizzing and ziffing. [Some people think that the word 'reverse' might be better but no one uses it for that purpose.]

Answer these questions on your work sheet.

(1) Because (42, 1806) belongs to the singulary operation ziffing, we can correctly conclude that (___?___, 42) belongs to fizzing, the inverse of ziffing.

(2) If (342, 18) belongs to fizzing, (18, ___?___) must belong to ziffing, the inverse of fizzing.

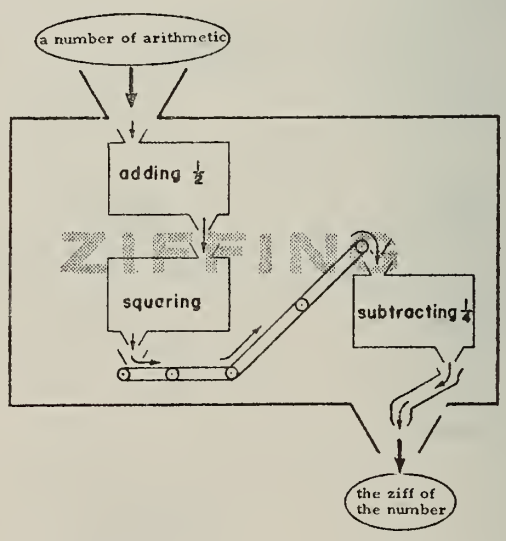
(3) If (9, 90) belongs to ziffing, (___?___, ___?___) must belong to fizzing.

(4) $(\frac{1}{2}, \frac{3}{4})$ belongs to ziffing and $(\frac{3}{4}, \frac{1}{2})$ belongs to fizzing; so, the ziff of $\frac{1}{2}$ is ___?___, and the fizz of $\frac{3}{4}$ is ___?___.

(5) Ziffing undoes what fizzing did; so, if we fizz 6 and ziff the result, we will get ___?___, the number we started with.

To satisfy the student's curiosity (and, possibly, suspicions) concerning ZIFFING, we use a "complex" function machine to show how to find the ordered pairs which belong to ZIFFING.

[Part III] [Page 34]



With this arrangement of "machines", we see, for example, that the ziff of $\frac{3}{2}$ is $\frac{15}{4}$,

since $\frac{3}{2} + \frac{1}{2} = 2$,

2 squared = 4,

and $4 - \frac{1}{4} = \frac{15}{4}$.

Now do the following exercises, and write the answers on your work sheet.

(1) Use the method shown above to find the ziff of $3\frac{1}{2}$.

(2) Compute the ziff of 9.5.

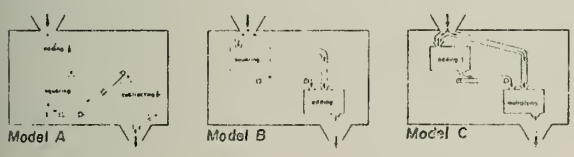
Of course the picture on the foregoing page is just one "model" of the ZIFFING machine. There are at least two other models of interest. The student is given the opportunity to test each model. A pattern sentence is displayed for each model to show how that model computes the ziff of a given number. We then take advantage of this situation to employ the available field axioms to show that, for a given number of arithmetic, ziffing in the Model C way is equivalent to ziffing in the Model B way. [There is also a proof that the Model A ziff of 111 is equal to the Model B ziff of 111. This, being a more difficult proof than the ones on pages 40 and 41, is found in a <u>voluntary</u> branch.]

Finally, on page 42, we point out that the proofs on pages 40 and 41 are not dependent upon the numbers chosen.

[Part 111] [Page 38]

As you have seen, we have claimed that there are three models of a ziffing machine.

ZIFFING



Model A	$z(_) = s(_ + \frac{1}{2}) - \frac{1}{4}$
Model B	$z(_) = s(_) + _$
Model C	$z(_) = (_ + 1) \times _$

Do these exercises on your work sheet.

(1) Use all three models to compute $z(2)$.

(2) Pick a number of arithmetic. Try all three models to compute the ziff of your number.

[Part 111] [Page 39]

Check your answers.

Model A	$z(_) = s(_ + \frac{1}{2}) - \frac{1}{4}$
Model B	$z(_) = s(_) + _$
Model C	$z(_) = (_ + 1) \times _$

(1) Model A computation:

$$z(2) = s(2 + \tfrac{1}{2}) - \tfrac{1}{4} = 2\tfrac{1}{2} - \tfrac{1}{4} = 6\tfrac{1}{4} - \tfrac{1}{4} = \underline{6}$$

Model B computation:

$$z(2) = s(2) + 2 = 2 \times 2 + 2 = 4 + 2 = \underline{6}$$

Model C computation:

$$z(2) = (2 + 1) \times 2 = 3 \times 2 = \underline{6}$$

(2) We are ABSOLUTELY certain that if you did your arithmetic correctly, you obtained the same result using all three models. [Consider yourself correct only if you <u>did</u> get the same result for all three models.]

Record your results.

[Part III] [Page 40]

Model B	z(_) = s(_) + _
Model C	z(_) = (_ + 1) × _

Let's see why we can be ABSOLUTELY certain that, no matter what number of arithmetic someone might choose, the Model C way of computing its ziff gives the same result as the Model B way.

Suppose that you pick an outlandish number, say, 1776 and get ready to compute its ziff the Model C way and the Model B way.

Model C ziff	Model B ziff
(1776 + 1) × 1776	s(1776) + 1776

Our claim is that the Model C ziff is the same as the Model B ziff. Of course, one way to check this is to carry out the computations. There is an easier way, a way which makes use of our basic principles.

Fill the blanks on your work sheet.

(1776 + 1) × 1776 = (1776 × 1776) + (1 × 1776) [?]
= (1776 × 1776) + (1776 × 1) [?]
= (1776 × 1776) + 1776 [?]
= s(1776) + 1776 [meaning of 's']

Hence, (1776 + 1) × 1776 = s(1776) + 1776

In other words, the Model C ziff of 1776 equals the Model B ziff of 1776.

[Part III] [Page 41]

Check your answers.

(1776 + 1) × 1776 = (1776 × 1776) + (1 × 1776) [dpma]
= (1776 × 1776) + (1776 × 1) [cpm]
= (1776 × 1776) + 1776 [pm1]
= s(1776) + 1776 [meaning of 's']

Hence, (1776 + 1) × 1776 = s(1776) + 1776.

In other words, the Model C ziff of 1776 equals the Model B ziff of 1776.

Record your results.

* * *

On your work sheet, write a proof whose conclusion is that the Model C ziff of 99 equals the Model B ziff of 99.

(99 + 1) × 99 = (99 × 99) + (1 × 99) [?]
= ? + ? [cpm]
= ? + ? [pm1]
= s(?) + ? [meaning of 's']

Hence, (99 + ?) × ? = s(?) + ? .

In other words, the Model C ziff of ? equals the Model B ziff of ? .

[Part III] [Page 42]

Check your answers.

The Model C ziff of 99 equals the Model B ziff of 99.

(99 + 1) × 99 = (99 × 99) + (1 × 99) [dpma]
= (99 × 99) + (99 × 1) [cpm]
= (99 × 99) + 99 [pm1]
= s(99) + 99 [meaning of 's']

Hence, (99 + 1) × 99 = s(99) + 99.

In other words, the Model C ziff of 99 equals the Model B ziff of 99.

Record your results.

* * *

It is easy to see from the pattern of the proofs for 1776 and 99 how to prove that the Model C and Model B ziffs are the same for any number of arithmetic. The fact that the Model C ziff of any number is the same as its Model B ziff is a consequence of the three basic principles for numbers of arithmetic:

(1) the ____?____ principle for ____?____ over addition

(2) the ____?____ principle for ____?____

(3) the principle for ____?____ by 1

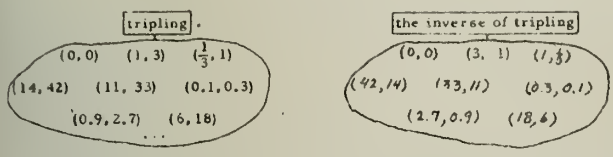
Fill the blanks on your work sheet.

In preparation for Parts 112 and 114.5, we develop the idea that a given list of pairs may have several names.

[Part 111] [Page 53]

Check your answers.

The correct pairs are listed below.



Record your results.

* * *

THE INVERSE OF TRIPLING is a singulary operation which when applied to 3, yields 1. When we operate on 18 with the inverse of tripling, the result is 6.

Do you recognize this singular[illegible]tion? What is another name for the inverse of tripling?

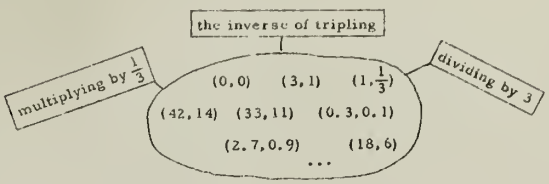
Write the answer on your work sheet.

[Part 111] [Page 54]

Check your answer.

The inverse of tripling is also called:

multiplying by $\frac{1}{3}$ or: *dividing by 3*



Record your result. You were correct if you wrote either or both of the answers given above.

* * *

Answer these questions.

(1) Dividing by 3 is the inverse of tripling. This means that dividing by 3 undoes what __?__ did.

(2) __?__ is the inverse of multiplying by $\frac{1}{3}$. Thus, since (6, 2) belongs to multiplying by $\frac{1}{3}$, (2, 6) belongs to __?__.

(3) Bill picks a number and triples it. Then, he applies the inverse of tripling to the result. The final result should be the number he __?__ with.

We use the idea of "many names for a given list" to lead the students to discover that

(1) for each nonzero number of arithmetic, 'dividing by that number' means the same thing as 'the inverse of multiplying by that number' (pages 57-61), and

(2) for each number of arithmetic, 'subtracting that number' means the same thing as 'the inverse of adding that number' (pages 63-65).

[Part 111] [Page 57]

Check your answers.

(1) Start with a number, say, 2. Apply a singulary operation to it, say, tripling. This singulary operation takes us from 2 to 6.

(2) Now, let's undo the result of this operation. In other words, let's perform the *inverse* of tripling.

(3) To undo the result of tripling 2 means to get back to 2.

(4) So, the *inverse* of tripling is the singulary operation which takes us from 6 to 2.

(5) Therefore, since tripling contains (2, 6), the pair (6, 2) must belong to the *inverse* of tripling.

Record your results.

* * *

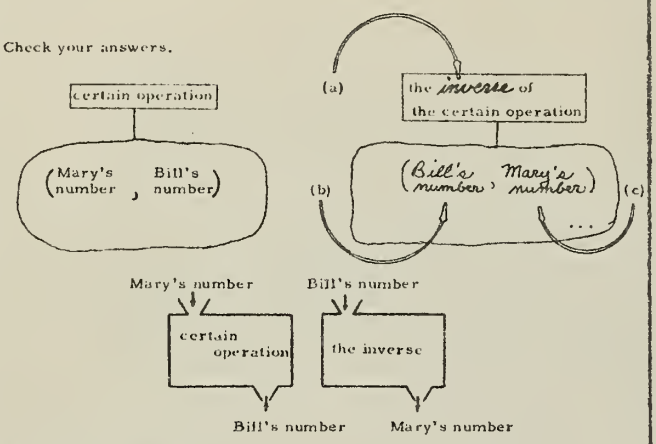
Mary and Bill pick numbers. Let's say that if you perform a certain operation on Mary's number, the result is Bill's number.

So, if you perform the __?__ of this operation on __?__ number, you get __?__ number.

Write the answers on your work sheet.

[Part III] [Page 58]

Check your answers.



Record your results.

* * *

Multiplying by 5 is a singulary operation which when applied to 1, gives 5. When applied to 6, multiplying by 5 gives 30.

Fill the blanks on your work sheet to make true sentences.

(1) When multiplying by 5 is applied to 11, the result is _?_.

(2) If there is an operation which undoes what multiplying by 5 did, then that operation applied to 55 would give _?_.

[Part III] [Page 59]

Check your answers.

(1) When multiplying by 5 is applied to 11, the result is 55.

(2) If there is an operation which undoes what multiplying by 5 did then that operation applied to 55 would give 11.

Record your results.

* * *

A singulary operation which undoes what multiplying by 5 did would be the inverse of multiplying by 5.

Fill the blanks.

(1) Any operation which undues what multiplying by 5 did is the _?_ of multiplying by 5.

(2) When we apply the inverse of multiplying by 5 tu a number, we get a second number, and it is always true that

this second number ^ _?_ = the first number.

[Part III] [Page 60]

Check yuur answers.

(1) Any operatiun which undoes what multiplying by 5 did is the inverse of multiplying by 5.

(2) When we apply the inverse of multiplying by 5 to a number, we get a second number, and it is always true that this second number ^ 5 = the first number.

Record your results.

* * *

The operation which is the inverse of multiplying by 5, we call:

DIVIDING BY 5

[The marks 'dividing by ___' mean the same thing as the marks 'the inverse of multiplying by ___'.]

Fill the blanks.

(1) Dividing by 5 is the operation which _?_ what _?_ by 5 did.

(2) If we multiply a number by 0.717 and then perform the inverse of multiplying by _?_, we will get back to the number we started with. The operation which is the inverse of multiplying by 0.717 is called '_?_ by 0.717'.

(3) By 'dividing by $\frac{1}{9}$', we mean nothing more than the operation which undoes what _?_ by _?_ did.

(4) By 'dividing by 6.7', we mean nothing more than the uperation which is the inverse of _?_ by _?_.

[Part III] [Page 61]

Check your answer.

(1) Dividing by 5 is the operation which undoes what multiplying by 5 did.

(2) If we multiply a number by 0.717 and then perform the inverse of multiplying by 0.717, we will get back to the number we started with The operation which is the inverse of multiplying by 0.717 is called dividing by 0.717'.

(3) By 'dividing by $\frac{1}{9}$', we mean nothing more than the operation which undoes what multiplying by 1/9 did.

(4) By 'dividing by 6.7', we mean nothing more than the operation which is the inverse of multiplying by 6.7.

Recurd your results.

* * *

Du these problems.

(1) (_?_ + 4) - 4 = 7

(2) (_?_ + 9) - 9 = 20

(3) (_?_ + 51) - 51 = 9

(4) (_?_ + 79) - 79 = 8423

(5) (8 + 7) - 7 = _?_

(6) (16 + 9) - 8 = _?_ [Careful!]

(7) (57 + 24) - _?_ = 57

(8) (24 + 57) - _?_ = 57

[Part 111] [Page 63]

Check your answers.

(1) (68 + 25) - 25 = 68 (3) (481 + 184) - 184 = 481

In each of exercises (2), (4), (5), and (6), you would be correct if you put a numeral for any number in the first blank, and a numeral for that same number in the second blank.

Record your results.

* * *

Here is a list of some pairs belonging to the singularly operation ADDING 6.

adding 6

(0, 6) (4, 10)
(3, 9) (9, 15) ($\frac{1}{4}$, $6\frac{1}{4}$) (0.1, 6.1)
(2, 8) . . . (11, 17)

Assume that ADDING 6 has an inverse. On your work sheet list the corresponding pairs which belong to the operation which undoes what adding 6 did.

[Part 111] [Page 64]

Check your answers. The correct pairs are listed below.

the inverse of adding 6

(6, 0) (10, 4)
(9, 3) (15, 9) ($6\frac{1}{4}$, $\frac{1}{4}$) (6.1, 0.1)
(8, 2) (17, 11)

Record your results.

* * *

Study the list above. What is another name for THE INVERSE OF ADDING 6?

Write the answer on your work sheet.

[Part 111] [Page 65]

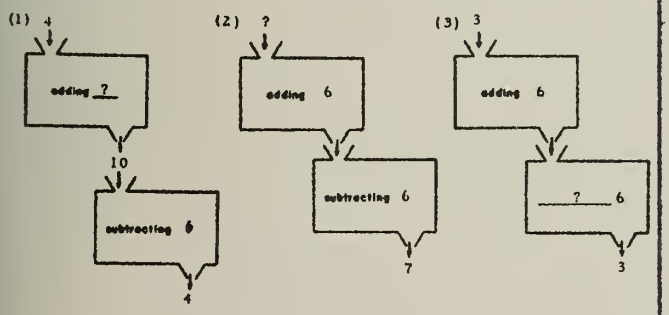
Check your answer.

The inverse of adding 6 is subtracting 6.

Record your result.

* * *

We can easily show that SUBTRACTING 6 undoes what ADDING 6 did by using the "machine" notation. Fill the blanks on your work sheet.



(4) ? 6 is the inverse of adding 6.

(5) ? 14 is the inverse of adding 14.

(6) The inverse of adding 4 is ? ?.

(7) The inverse of SUBTRACTING 6 is ? ?.

In the concluding pages of Part 111, the students get practice in operations on real numbers. In particular, they contact SQUARING (pages 72-76)--an operation which has no inverse.

[Part 111] [Page 72]

Check your answers.

doubling	the inverse of doubling
(⁺823, ⁺1646)	(⁺1646, ⁺823)
(⁻406, ⁻812)	(⁻812, ⁻406)
(⁺51, ⁺102)	(⁺102, ⁺51)
(⁻812, ⁻1624)	(⁻1624, ⁻812)
(⁺102, ⁺204)	(⁺204, ⁺102)
([redacted], ⁺16)	(⁺16, ⁺8)
([redacted], ⁻16)	(⁻16, ⁻8)
...	...

Did you get all seven pairs? You might have worried about the ink blots, but they shouldn't have stopped you. From what you know about DOUBLING, you can tell that there is only one number whose double is, say, ⁺16. So, you can be sure about what is hidden by the ink blot in '([redacted], ⁺16)'.

Record your results.

* * *

Here is a list of seven of the pairs in SQUARING.

SQUARING		
(⁺823, ⁺677329)	(⁻406, ⁺164836)	(⁺51, ⁺2601)
(⁻812, ⁺659344)	(⁺102, ⁺10404)	([redacted], ⁺16)
([redacted], 0)	...	

Assume that SQUARING has an inverse--that is, assume that there is a singulary operation which undoes what SQUARING did. Then, on your work sheet, list exactly seven pairs which correspond to the ones given above.

[Part 111] [Page 73]

Check your answers.

squaring	assumed the ^ inverse of squaring
(⁺823, ⁺677329)	(⁺677329, ⁺823)
(⁻406, ⁺164836)	(⁺164836, ⁻406)
(⁺51, ⁺2601)	(⁺2601, ⁺51)
(⁻812, ⁺659344)	(⁺659344, ⁻812)
(⁺102, ⁺10404)	(⁺10404, ⁺102)
([redacted], ⁺16)	(⁺16,) ⁺4?
([redacted], 0)	(0, 0) ⁻4?
...	...

Here is a case in which the ink blot makes it impossible to carry out the job. Of course, in the case of '([redacted], 0)', you can tell what is hidden under the ink blot because there is only one number whose square is 0. But, in the case of '([redacted], ⁺16)', you just can't tell what's under the blot; there are TWO numbers which have ⁺16 as there square. [If you realized this about the '([redacted], ⁺16)', count yourself correct for that item. Otherwise, count yourself wrong.]

Record your results.

* * *

Now, answer these questions.

(1) Bill picks a real number and doubles it. The result is ⁺36. Tell EXACTLY what number Bill started with.

(2) Mary picks a real number and squares it. The result is ⁺36. Tell EXACTLY what number Mary started with.

Assume, again, that there is a singulary operation which is the inverse of squaring. Here is a list of some of its pairs.

assumed
the ^ inverse of squaring

($^{+}81$, $^{+}9$) ($^{+}81$, $^{-}9$) ($^{+}64$, $^{+}8$)

($^{+}64$, $^{-}8$) (0, 0) ($^{+}25$, $^{-}5$) ($^{+}25$, $^{+}5$)

($^{+}225$, $^{-}15$) ($^{+}225$, $^{+}15$) ($^{+}\frac{1}{4}$, $^{-}\frac{1}{2}$)

($^{+}\frac{1}{4}$, $^{+}\frac{1}{2}$) . . .

(*) As we pointed out on Page 2, an operation gives you exactly one number whenever it is applied.

(1) Does the description in (*) fit what we are calling 'the assumed inverse of squaring'? [Answer on your work sheet.]

(2) Exactly what number do you get if you try to apply 'the assumed inverse of squaring' to $^{+}81$?

(3) Do you think that there is a singulary operation which can be called 'the inverse of squaring'?

Check your answers.

Description of an operation:

An operation gives you exactly one number whenever it is applied.

(1) No, this description does not fit what we are calling 'the assumed inverse of squaring.

(2) For example, if we try to apply it to $^{+}81$, we don't come out with exactly one number. Answer: Can't tell exactly [or something like this]

(3) No, there is no singulary operation which can properly be called, 'the inverse of squaring'. Our assumption that there is such an operation is wrong.

~~assumed~~
~~the ^ inverse of squaring~~

($^{+}81$, $^{+}9$) ($^{+}81$, $^{-}9$) ($^{+}64$, $^{+}8$)

($^{+}64$, $^{-}8$) (0, 0) ($^{+}25$, $^{-}5$) ($^{+}25$, $^{+}5$)

($^{+}225$, $^{-}15$) ($^{+}225$, $^{+}15$) ($^{+}\frac{1}{4}$, $^{-}\frac{1}{2}$)

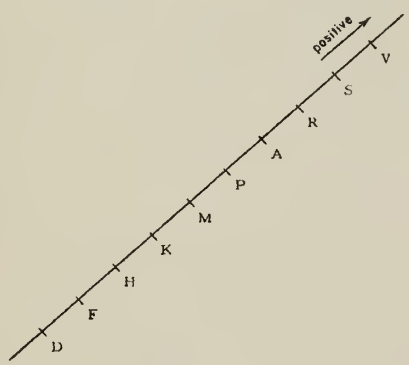
($^{+}\frac{1}{4}$, $^{+}\frac{1}{2}$) . . .

PART 112

The road maps, introduced in Parts 102 and 103, provide the physical situation from which, on pages 1-5 of Part 112, we extract the operation OPPOSITING.

[Part 112] [Page 1]

Here is a map of a road.



The markers are 1 mile apart. Let's agree that the direction from D to V is the positive direction. Then, for example,

a trip from F to M is measured by $^{+}3$,

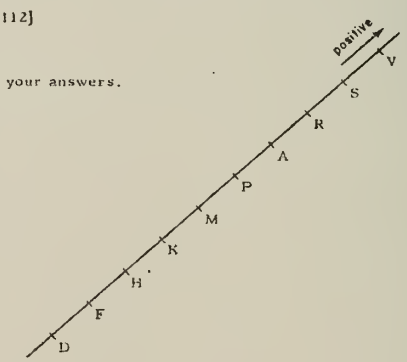
and the return trip from M to F is measured by $^{-}3$.

On your work sheet complete the following table. [The first row is a sample.]

	Trip	Measure	Return Trip	Measure
	H to P	$^{+}3$	P to H	$^{-}3$
(1)	S to M	?	M to S	?
(2)	F to A	?	A to ?	?
(3)	V to ?	$^{-}2$	? to V	$^{+}2$
(4)	H to R	?	R to ?	?
(5)	M to ?	?	? to M	$^{+}4$

[Part 112] [Page 2]

Check your answers.



	Trip	Measure	Return Trip	Measure
(1)	S to M	$^{-}4$	M to S	$^{+}4$
(2)	F to A	$^{+}5$	A to F	$^{-}5$
(3)	V to R	$^{-}2$	R to V	$^{+}2$
(4)	H to R	$^{+}5$	R to H	$^{-}5$
(5)	M to D	$^{-}4$	D to M	$^{+}4$

Record your results.

* * *

In each of the following exercises, you are given a first real number. Consider this as the measure of a trip along a road. Then, give a second real number which measures the return trip.

(1) ($^{-}2$, ?) (2) ($^{+}3$?) (3) ($^{-}6$, ?)

(4) ($^{+}7$, ?) (5) ($^{-}7$, ?) (6) (0, ?)

(7) ($^{-}2.1$, ?) (8) ($\frac{^{+}8}{3}$, ____) (9) ($\frac{^{-}8}{3}$, ?)

[Part 112] [Page 3]

Check your answers.

Pairs of numbers for which the first measures a trip and the second measures the return trip.

(1) ($^-2$, $^+2$) (2) ($^+3$, $^-3$) (3) ($^-6$, $^+6$)
(4) ($^+7$, $^-7$) (5) ($^-7$, $^+7$) (6) (0, 0)
(7) ($^-2.1$, $^+2.1$) (8) ($\frac{^+8}{5}$, $\frac{^-8}{5}$) (9) ($\frac{^-8}{3}$, $\frac{^+8}{3}$)

Record your results.

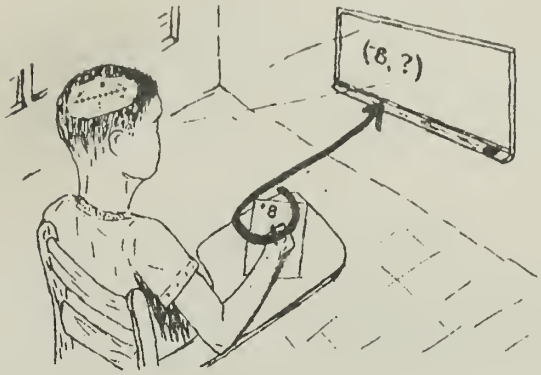
* * *

Here is a picture of a person doing a problem such as those you have just done.

Write the answer to his problem on your work sheet.

[Part 112] [Page 3a]

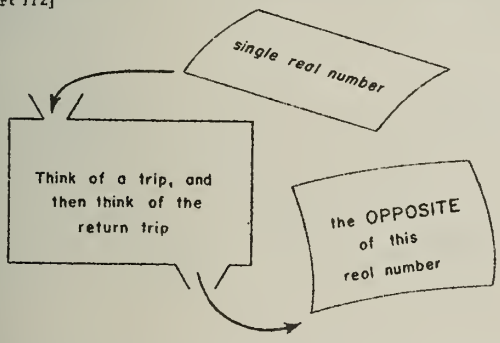
Check your answer.



Record your result.

Go to PAGE 4.

[Part 112] [Page 4]



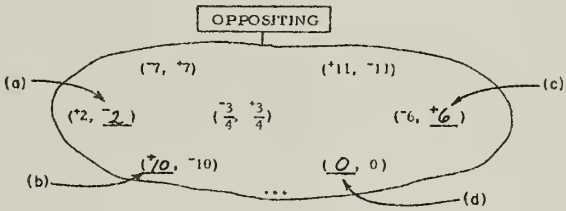
When you start with a real number and then come out with the OPPOSITE of this real number, you are performing a singulary operation. This singulary operation is called

OPPOSITING

($^-7$, $^+7$) ($^+11$, $^-11$)
($^+2$, ?) ($\frac{^-3}{4}$, $\frac{^+3}{4}$) ($\frac{^+3}{4}$, $\frac{^-3}{4}$) ($^-6$, ?)
(?, $^-10$) (?, 0)
...

Give the missing numbers on your work sheet. [Pronounce 'oppositing' so that it rhymes with 'baby-sitting'.]

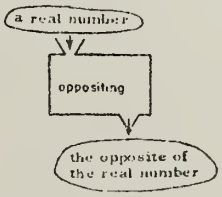
[Part 112] [Page 5]



Record your results.

* * *

As you have seen, OPPOSITING is a singulary operation on real numbers.



Answer these questions on your work sheet.

(1) The real number ? is the opposite of $^-4$.
(2) The opposite of $^-8$ is ?.
(3) ? is the opposite of $^+6$.
(4) The opposite of ? is $^+2$.
(5) ? ? ? $^+6$ is $^-6$.
(6) $^+9$ is ? ? ? $^-9$

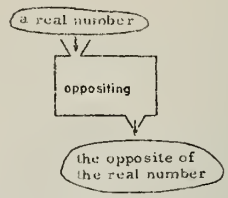
The introduction of function notation (page 6) is quite naturally motivated by a desire to have a shorter way of writing:

the opposite of x

We have, moreover, other reasons for introducing function notation at this point. It allows us, after the discovery exercises on pages 7 and 8, to give a compact pattern sentence for THE PRINCIPLE OF OPPOSITES on page 9.

[Part 112] [Page 6]

Check your answers.



(1) The real number $^{+}4$ is the opposite of $^{-}4$.

(2) The opposite of $^{-}8$ is $^{+}8$.

(3) $^{-}6$ is the opposite of $^{+}6$.

(4) The opposite of $^{-}2$ is $^{+}2$.

(5) The opposite of $^{+}6$ is $^{-}6$.

(6) $^{+}9$ is the opposite of $^{-}9$.

Record your results.

* * *

Instead of writing, say:

the opposite of $^{-}8$

let's agree to write the shorter symbol:

$\mathcal{O}(^{-}8)$ [Read this as 'the opposite of negative eight'.]

So, for example, we can write:

$\mathcal{O}(^{+}5) = {}^{-}5$, $\mathcal{O}(^{-}6) = {}^{+}6$, and: $\mathcal{O}(0) = 0$

Compute. [Write the answers on your work sheet.]

(1) $\mathcal{O}(^{-}11) = \underline{?}$ (2) $\mathcal{O}(^{+}9) = \underline{?}$

(3) $\mathcal{O}(\underline{?}) = {}^{+}6$ (4) $\mathcal{O}(^{+}6) + \mathcal{O}(^{-}2) = \underline{?}$

[Part 112] [Page 7]

Check your answers.

(1) $\mathcal{O}(^{-}11) = \underline{^{+}11}$ (2) $\mathcal{O}(^{+}9) = \underline{^{-}9}$

(3) $\mathcal{O}(\underline{^{-}6}) = {}^{+}6$ (4) $\underbrace{\mathcal{O}(^{+}6)}_{^{-}6} + \underbrace{\mathcal{O}(^{-}2)}_{^{+}2} = \underline{^{-}4}$

Record your results.

* * *

Answer these questions on your work sheet

(1) $\mathcal{O}(^{+}5) + \mathcal{O}(^{-}2)$
$= {}^{-}5 + \underline{?}$
$= \underline{?}$

(2) $\mathcal{O}(^{-}7) + \mathcal{O}(^{-}3) = \underline{?}$

(3) $\mathcal{O}(^{+}7) + \mathcal{O}(^{+}3) = \underline{?}$

(4) $^{+}5 + \mathcal{O}(^{+}5) = \underline{?}$

(5) $^{-}7 + \mathcal{O}(^{-}7) = \underline{?}$

(6) $\underline{?} + \mathcal{O}(^{+}8) = 0$

(7) $^{-}42 + \mathcal{O}(\underline{?}) = 0$

[Part 112] [Page 8]

Check your answers.

(1) $\mathcal{O}(^{+}5) + \mathcal{O}(^{-}2)$
$= {}^{-}5 + \underline{^{+}2}$
$= \underline{^{-}3}$

(2) $\mathcal{O}(^{-}7) + \mathcal{O}(^{-}3) = \underline{^{+}10}$

(3) $\mathcal{O}(^{+}7) + \mathcal{O}(^{+}3) = \underline{^{-}10}$

(4) $^{+}5 + \mathcal{O}(^{+}5) = \underline{0}$

(5) $^{-}7 + \mathcal{O}(^{-}7) = \underline{0}$

(6) $\underline{^{+}8} + \mathcal{O}(^{+}8) = 0$

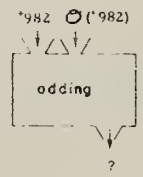
(7) $^{-}42 + \mathcal{O}(\underline{^{-}42}) = 0$

Record your results.

* * *

Answer these questions on your work sheet.

(1)

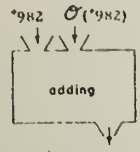


(2) $^{-}3465 + \mathcal{O}(^{-}3465) = \underline{?}$

[Part 112] [Page 9]

Check your answers.

(1)



+982 + 𝒪(+982) = 0

(2) −3465 + 𝒪(−3465) = 0

Record your results.

* * *

As you have probably discovered, one of the important principles about the operation OPPOSITING is that any real number plus its opposite is 0. This principle is called:

THE PRINCIPLE OF OPPOSITES

Pattern: ~~~ + 𝒪(~~~) = 0

Complete the following sentences on your work sheet.

(1) An instance of the principle of opposites is:

−12 + 𝒪(?) = 0

(2) The sentence '+628 + 𝒪(+628) = 0' is an instance of ? ? ? ?.

In Part 111 we agreed that:

> for each number of arithmetic,
> subtracting that number
> is the same thing as
> the inverse of adding that number

We are now ready to motivate a definition of:

subtracting the real number x

on the basis of this agreement. Our first move is to have the student discover that for each real number x,

adding 𝒪(x)

is the same thing as

the inverse of adding x.

This fact is first suggested in the exercises on pages 12-13.

[Part 112] [Page 11]

Check your answers.

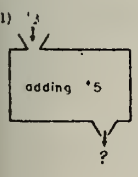
(1) +7 + 𝒪(−3) = +7 + +3 = +10
(2) +1 + 𝒪(+8) = +1 + −8 = −7
(3) +4 + 𝒪(−5) = +9
(4) −6 + 𝒪(+3) = −9
(5) −7 + 𝒪(−11) = +4
(6) +9 + 𝒪(−4) = +13
(7) +12 + −2 = +10
(8) +12 + 𝒪(+2) = +10

Record your results.

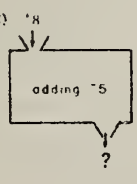
* * *

Answer these questions.

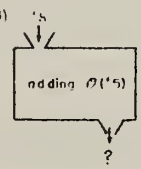
(1)



(2)



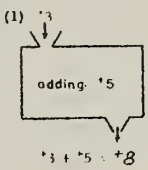
(3)



[Part 112] [Page 12]

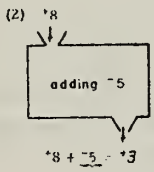
Check your answers.

(1)



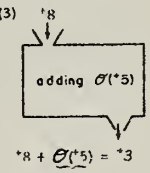
+3 + +5 = +8

(2)



+8 + −5 = +3

(3)



+8 + 𝒪(+5) = +3

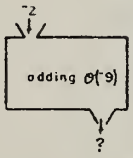
Record your results.

* * *

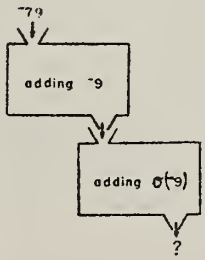
Answer these questions.

(1)

(2)

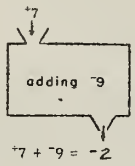


(3)



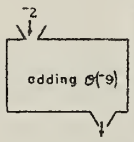
Check your answers.

(1)



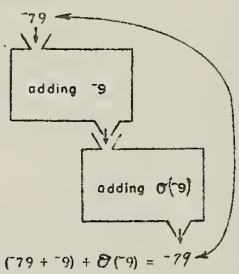
$^{+}7 + {}^{-}9 = {}^{-}2$

(2)



$^{-}2 + \mathcal{O}(^{-}9) = {}^{+}7$

(3)



$(^{-}79 + {}^{-}9) + \mathcal{O}(^{-}9) = {}^{-}79$

Record your results.

* * *

Answer these questions.

(1) $(^{+}8 + {}^{-}6) + {}^{+}6 =$?

(2) $(^{+}8 + {}^{-}6) + \mathcal{O}(^{-}6) =$?

(3) $(^{-}5 + {}^{+}7) + \mathcal{O}(^{+}7) =$?

(4) $(^{+}1776 + {}^{-}1962) + \mathcal{O}(^{-}1962) =$?

(5) $(^{-}1492 + {}^{-}1999) + \mathcal{O}($? $) = {}^{-}1492$

On pages 16-19, we use lists to suggest again the relationship between

the inverse of adding x

and

adding (x).

Check your answers.

To prove that $(^{+}718 + {}^{-}461) + \mathcal{O}(^{-}461) = {}^{+}718$:

$(^{+}718 + {}^{-}461) + \mathcal{O}(^{-}461) = {}^{+}718 + [{}^{-}461 + \mathcal{O}(^{-}461)]$ [apa]

$= {}^{+}718 + 0$ [principle of opposites]

$= {}^{+}718$ [pal]

So, $(^{+}718 + {}^{-}461) + \mathcal{O}(^{-}461) = {}^{+}718$.

Record your results.

* * *

Here is a list of seven pairs for the singulary operation ADDING 2.

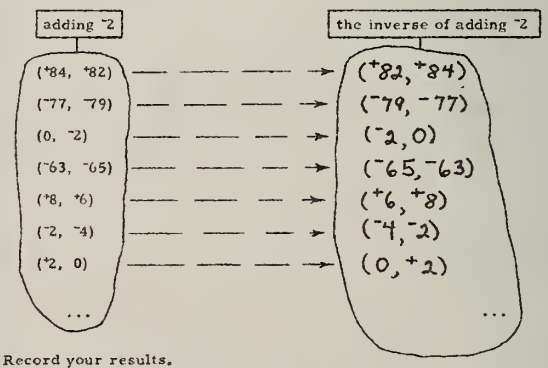
adding ⁻2

(⁺84, ⁺82) (77, ⁻79)
(0, ⁻2)
(⁻63, ⁻65)
(⁺8, ⁺6)
(⁻2, ⁻4) (⁺2, 0) ....

This singulary operation has an INVERSE -- that is, there is a singulary operation which undoes what adding ⁻2 did.

On your work sheet give seven pairs which belong to THE INVERSE OF ADDING ⁻2 and which correspond to those given above.

Check your answers.



Record your results.

* * *

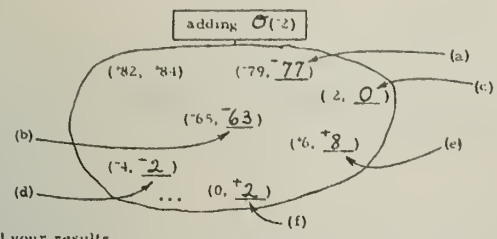
Find the missing numbers for this list of seven pairs for the singulary operation ADDING $\mathcal{O}(^{-}2)$.

adding $\mathcal{O}(^{-}2)$

(⁺82, ⁺84) (⁻79, ?)
(⁻2, ?)
(⁻65, ?)
(⁺6, ?)
(⁻4, ?)
(0, ?)

[Part 112] [Page 3]

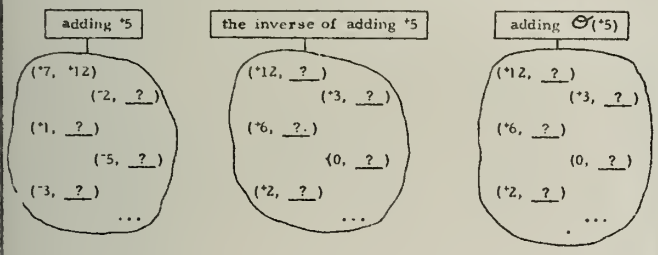
Check your answers.



Record your results.

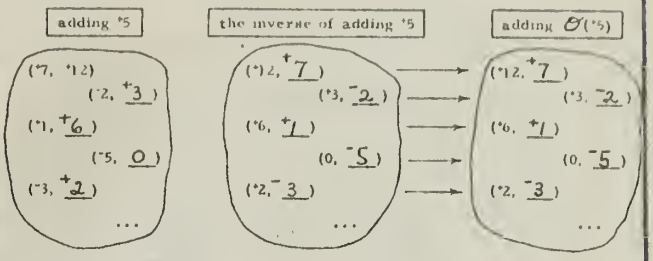
* * *

Find the missing numbers for these lists.



[Part 112] [Page 19]

Check your answers.



Record your results.

* * *

Answer these questions on your work sheet.

(1) Consider the singulary operation ADDING +5.
This operation takes you, for example, from +8 to ___?___.

(2) THE INVERSE OF ADDING +5 is a singulary operation which undoes what ADDING +5 does. So, the inverse of adding +5 takes you from +13 back to ___?___.

(3) The operation ADDING $\mathcal{O}$(+5) also takes you from +13 to ___?___.

The proof on pages 20-22 emphasizes what is meant by 'the inverse of an operation'. It lends confidence to the conclusion on page 22.

[Part 112] [Page 20]

Check your answers.

(1) The singulary operation ADDING +5 takes you from +8 to ___+13___.

(2) Because THE INVERSE OF ADDING +5 undoes what ADDING +5 does, it takes you from +13 back to ___+8___.

(3) The singulary operation ADDING $\mathcal{O}$(+5) also takes you from +13 to ___+8___.

Record your results.

* * *

Now, in view of what you have done so far, you have probably guessed that

THE INVERSE OF ADDING +5 is the same thing as ADDING $\mathcal{O}$(+5).

This is the case. Let's see if we can find out why. [Fill the blanks on your work sheet.]

Pick some "outlandish" real number, say, -863.

(1) If we apply ADDING +5 to -863, we get (-863 + ___?___).

(2) If we did apply THE ___?___ OF ___?___ ___?___ to this result, we would get back ___?___.

(3) But, let's apply the singulary operation ADDING $\mathcal{O}$(+5) to (-863 + +5). What we get is (-863 + +5) + $\mathcal{O}$(___?___).

(4) We can easily prove that this is -863. Here's how.

(-863 + +5) + $\mathcal{O}$(+5) = -863 + (___?___ + $\mathcal{O}$(+5)) [___?___]
= -863 + ___?___ [___?___]
= -863 [___?___]

So, (-863 + +5) + $\mathcal{O}$(___?___) = -863.

(5) So, if we apply ADDING +5 to -863 and then apply ADDING $\mathcal{O}$(___?___) to the result, we get back -863. Thus, in the case of -863, ADDING $\mathcal{O}$(___?___) undid what ADDING +5 did.

[Part 112] [Page 21]

Check your answers.

Pick some "outlandish" number, say, $^-863$.

(1) If we apply the singular operation ADDING $^+5$ to this number, we get ($^-863$ + $^+5$).

(2) If we did apply THE INVERSE OF ADDING $^+5$ to this result, we would get back $^-863$.

(3) But, let's apply the singular operation ADDING $\mathcal{O}(^+5)$ to ($^-863$ + $^+5$). What we get is ($^-863$ + $^+5$) + $\mathcal{O}$($^+5$).

(4) We can easily prove that this is $^-863$. Here's how.

($^-863$ + $^+5$) + $\mathcal{O}$($^+5$) $^-863$ + ($^+5$ + $\mathcal{O}$($^+5$)) [app]

$^-863$ + 0 [principle of opposites]

= $^-863$ [zero]

So, ($^-863$ + $^+5$) + $\mathcal{O}$($^+5$) = $^-863$.

(5) So, if we apply ADDING $^+5$ to $^-863$ and then apply ADDING $\mathcal{O}$($^+5$) to the result, we get back $^-863$. Thus, in the case of $^-863$, ADDING $\mathcal{O}$($^+5$) undid what ADDING $^+5$ did.

Record your result.

* * *

Now, look carefully at the foregoing discussion. Notice that even though it refers to $^-863$, the number $^-863$ didn't get involved in any of the computations. If we pick any other real number and replace each of the '$^-863$'s by a numeral for that number, the resulting discussion would make perfectly good sense.

So, what we have really shown is that, no matter what real number you pick, applying ADDING $^+5$ to it, and then applying ADDING $\mathcal{O}$($^+5$) to the result, brings you back to the number you started with. In other words, applying ADDING $\mathcal{O}$($^+5$) is precisely the same thing as applying ___?___ ___?___ ___?___ ___?___ $^+5$.

[Fill the blanks on your work sheet.]

[Part 112] [Page 22]

Check your answers.

ADDING $\mathcal{O}$($^+5$)

is precisely the same thing as

the inverse of adding $^+5$.

Record your results.

* * *

If you look again at the discussion on Page 21, you will find that although it refers to $^+5$, it would make as much sense if it referred to any other real number. So, no matter what real number you might consider,

THE INVERSE OF ADDING THAT NUMBER

is precisely the same thing as

ADDING THE ___?___ OF THAT NUMBER.

Fill the blank on your work sheet.

On page 25 we recall our definition of:

subtracting x

for the numbers of arithmetic and use that idea to define:

subtracting x

for the real numbers. Coupling this agreement with the discovery just made, we are ready to state the principle which appears on page 28.

[Part 112] [Page 25]

Check your answers.

(1) [redacted] + $^+3$ = 9

12, because 9 + $\underbrace{\mathcal{O}(^+3)}_{\text{the inverse of adding } ^+3}$ = 9 + $^-3$ $^-12$

(2) [redacted] + 2 = $^+7$

$^+9$, because $^+7$ + $\underbrace{\mathcal{O}(2)}_{\text{the inverse of adding } ^-2}$ = $^+7$ + $^+2$ = $^+9$

(3) [redacted] + $^-6$ = 31

$^-25$, because $^-31$ + $\underbrace{\mathcal{O}(^-6)}_{\text{the inverse of adding } ^-6}$ = $^-31$ + 6 = $^-25$.

Record your results.

* * *

In an earlier part of the course you learned that for each number of arithmetic:

subtracting that number

means nothing more than:

the inverse of adding that number

Similarly, for example, when we say:

THE INVERSE OF ADDING $^+5$

we could more simply say:

SUBTRACTING $^+5$

Answer these questions.

(1) A simpler way of saying: THE INVERSE OF ADDING $^-3$ is to say: ___?___ ___?___

(2) The phrase 'SUBTRACTING $^+10$' means precisely the same thing as the phrase '___?___ ___?___ ___?___ ___?___ ___?___'.

[Part 112] [Page 26]

Check your answers.

(1) A simpler way of saying: 'THE INVERSE OF ADDING $^{+}3$ is to say: subtracting $^{-}3$

(2) The phrase 'SUBTRACTING $^{+}10$' means precisely the same thing as the phrase 'the inverse of adding $^{+}10$'.

Record your results.

* * *

Let us agree that for each real number 'subtracting that number' means the same thing as 'the inverse of adding that number'. Hence, any problem in which you need to subtract a real number is equivalent to a problem in applying the inverse of adding that real number.

Answer these questions on your work sheet.

(1) $^{+}10$ → subtracting $^{+}3$ → ?

(2) $^{+}10$ → the inverse of adding $^{+}3$ → ?

(3) $^{+}10$ → adding $\mathcal{O}(?)$ → $^{+}7$

[Part 112] [Page 27]

Check your answers.

(1) $^{+}10$ → subtracting $^{+}3$ → $^{+}7$

(2) $^{+}10$ → the inverse of adding $^{+}3$ → $^{+}7$

(3) $^{+}10$ → adding $\mathcal{O}(?)$ → $^{+}7$

Record your results.

* * *

By our agreement on Page 27, we know that a problem in applying the singulary operation SUBTRACTING A REAL NUMBER is nothing more than a problem in applying 'THE INVERSE OF ADDING THAT REAL NUMBER.

And, as you discovered earlier, a problem in applying THE INVERSE OF ADDING A REAL NUMBER is nothing more than a problem in applying the singulary operation ADDING THE OPPOSITE OF THAT REAL NUMBER.

So, we can conclude that, no matter what real number we might pick,

SUBTRACTING THAT NUMBER

is the same thing as

__?__ __?__ __?__ __?__ __?__ __?__ .

[Part 112] [Page 28]

Check your answers.

> For each real number,
>
> SUBTRACTING THAT NUMBER
>
> is the same thing as
>
> ADDING THE OPPOSITE OF THAT NUMBER.

Record your results.

* * *

Use the principle shown in the box above to change each of the following subtraction problems into an equivalent addition problem.

(1) $^{+}8$, subtract $^{+}5$ ⟶ $^{+}8$, add $\mathcal{O}(?)$

(2) $^{-}11$, subtract $^{+}3$ ⟶ $^{-}11$, add ? (?)

(3) $^{+}7$, subtract $^{-}5$ ⟶ ?, ? ? (?)

To break up the monotony of drill, we use some self-checking cross-number puzzles like the one on page 41. Try it!

Check your answers.

(1) $^{-}2 - {}^{-}19 = {}^{-}2 + {}^{+}19 = {}^{+}\underline{17}$

(2) $^{-}\underline{17} - {}^{-}7 = {}^{-}10$ [Check: $^{-}17 - {}^{-}7 = {}^{-}17 + {}^{+}7 = {}^{-}10$]

(3) $^{-}14 - {}^{+}\underline{6} = {}^{-}20$ [Check: $^{-}14 - {}^{+}6 = {}^{-}14 + {}^{-}6 = {}^{-}20$]

(4) $^{-}6 - {}^{+}16 = {}^{-}6 + {}^{-}16 = {}^{-}\underline{22}$

(5) $^{+}\underline{12} - {}^{+}8 = {}^{+}4$ [Check: $^{+}12 - {}^{+}8 = {}^{+}12 + {}^{-}8 = {}^{+}4$]

(6) $^{-}7 - {}^{-}\underline{17} = {}^{+}10$ [Check: $^{-}7 - {}^{-}17 = {}^{-}7 + {}^{+}17 = {}^{+}10$]

Record your results.

* * *

Fill in the blanks on your work sheet so that each sentence running horizontally or vertically will be true.

$^{-}6$	$-$	$^{+}4$	$=$	
$+$		$-$		$+$
	$+$		$=$	$^{+}4$
$=$		$=$		$=$
	$=$	$^{+}7$	$+$	

PART 114

In Part 114, we find use for zebra pages in doing drill.

Students, having seen two methods for simplifying complicated expressions like that on page 30, are sent (from page 33) to [Q I] of the zebra section beginning on page 38. Frames [Q I]-[Q10] provide a compact form for drill in this type of exercise.

Upon checking his answers to the problems in [Q10] on page 41, the student is either sent ahead to the zebra section beginning on page 42, or is sent back to frame [Q11] on page 39 for additional practice.

[Part 114] [Page 30]

Check your answers.

(1) $10 - -7 - -4 + -5 - 15$
$= 10 + - -7 + - -4 + -5 + -15$
$= 10 + 7 + 4 + -5 + -15$
$= 1$

(2) $-11 - 9 - -7 - -3$
$= -11 + -9 + - -7 + - -3$
$= -11 + -9 + 7 + 3$
$= -10$

* * *

Example. Simplify:

$$2 - 5 + 6 - 3 + 9 - 6 - 5$$

Solution.

METHOD I
[left-to-right]

$2 - 5 + 6 - 3 + 9 - 6 - 5$
$= 2 + -5 + 6 + -3 + 9 + -6 + -5$
$= -3 + 6 + -3 + 9 + -6 + -5$
$= 3 + -3 + 9 + -6 + -5$
$= 0 + 9 + -6 + -5$
$= 9 + -6 + -5$
$= 3 + -5$
$= -2$

METHOD II
[rearrange using the cpa and the apa]

$2 - 5 + 6 - 3 + 9 - 6 - 5$
$= 2 + -5 + 6 + -3 + 9 + -6 + -5$
$= (2 + 6 + 9) + (-5 + -3 + -6 + -5)$
$= 17 + -19$
$= -2$

Simplify, using either method. Write the answer on your work sheet.

$$3 - 5 + 6 - 7 + 9 + 8 - 3$$

Simplify this expression:
$-3 - 6 + 4 - 7 + 7 - 13$
Answer on your work sheet.
Turn to panel [A 1] on PAGE 39 to check your answer.

[Q 1]

$(-4 + -8 + -17) + (0 + 3 + 11 + 16)$
$= -29 + 30$
$= 1$
Record your result.

[A 3]

Simplify:
$2 - \frac{1}{2} + 6 + 1\frac{1}{2} - 7 - 3$
Answer on your work sheet.
Turn to panel [A 4] on PAGE 39.

[Q 4]

$\left(-\frac{1}{2} + -\frac{1}{4} + -5\frac{1}{2}\right) + \left(\frac{3}{4} + 3\frac{1}{4}\right)$
$= -6\frac{1}{4} + 4$
$= -2\frac{1}{4}$
Record your result.

[A 6]

Simplify:
$-10 + 17 - 3 + 10 + 7 + 3 + 1$
Answer on your work sheet.
Turn to panel [A 7] on PAGE 39.

[Q 7]

$10 + 17 + 3 + 10 + 7 + 3 + 1$
$= 51$
Record your result.

[A 9]

Simplify: [Work carefully on these.]
(1) $8\frac{1}{2} + 9 - 2\frac{1}{2} - 7\frac{1}{2}$ (2) $6\frac{1}{3} - 2\frac{1}{4} - 3\frac{1}{2} + 3\frac{1}{6} -$
(3) $4.83 - 10 - 3.8 + 7.9$
(4) $16.75 - 11.3 + 40.72 - 61.007$

[Answers on PAGE 41.] [Q 10]

$\left(-7\frac{1}{2} + -8\frac{1}{4} + -3\frac{3}{4}\right) + \left(2\frac{1}{4} + 1\frac{1}{2}\right)$
$= \left(-7\frac{1}{2} + -12\right) + 3\frac{3}{4}$
$= -19\frac{1}{2} + 3\frac{3}{4}$
$= -15\frac{3}{4}$
Record your result. [A 12]

Simplify:
$.1 - 0.2 + 0.3 - 0.6 + 0.9$
Answer on your work sheet.
Turn to panel [A 13] on PAGE 39.

[Q 13]

$(-3 + -6 + -7 + -13) + (4 + 7)$

$= -29 + 11$

$= -18$

Record your result. [A 1]

Simplify:

$3 - 7 + 3 + 7 - 5 + 12 - 3$

Turn to panel [A 2] on PAGE 40.

[Q 2]

$(3 + 3 + 7 + 12) + (-7 + -5 + -3)$

$= 25 + -15$

$= 10$

[A 2]

Simplify:

$-4 + 6 - 8 + 3 + 11 - 17 + 16$

Turn to panel [A 3] on PAGE 38.

[Q 3]

$\left(2 + 6 + 1\frac{1}{2}\right) + \left(-\frac{1}{2} + -7 + -3\right)$

$= 9\frac{1}{2} + -10\frac{1}{2}$

$= -1$

[A 4]

Simplify:

$7 - 3 + 3 - 8 + 17 + 1 + 6 + 4$

Turn to panel [A 5] on PAGE 40.

[Q 5]

$(7 + 17 + 1 + 6 + 4) + (-3 + -3 + -8$

$= 35 + -14$

$= 21$

[A 5]

Simplify:

$-\frac{1}{2} + \frac{3}{4} - \frac{1}{4} + 3\frac{1}{4} - 5\frac{1}{2}$

Turn to panel [A 6] on PAGE 38.

[Q 6]

$(17 + 10 + 7 + 3 + 1) + (-10 - 3)$

$= 38 + -13$

$= 25$

[A 7]

Simplify:

$0.5 + 0.3 - 0.8 - 0.7 - 0.2 + 1.2$

Turn to panel [A 8] on PAGE 40.

[Q 8]

$(0.8 + 0.3 + 1.2) + (-0.8 + -0.7 + -1.2)$

$= 2 + -1.7$

$= 0.3$

[A 8]

Simplify:

$10 + 17 + 3 + 10 + 7 + 3 + 1$

Answer on your work sheet.

Turn to panel [A 9] on PAGE 38.

[Q 9]

[A 10] on PAGE 41.

Simplify:

$-10 + 10 - 19 - 3 + 3 + 0 + 2 - 2$

Answer on your work sheet.

Turn to panel [A 11] on PAGE 40.

[Q 11]

$(-10 + -10 + -3 + -2) + (10 + 3 + 0 + 2)$

$= -25 + 15$

$= -10$

[A 11]

Simplify:

$-7\frac{1}{2} + 2\frac{1}{4} + 1\frac{1}{2} + 8\frac{1}{4} - 3\frac{3}{4}$

Turn to panel [A 12] on PAGE 38.

[Q 12]

$(0.1 + 0.3 + 0.9) + (-0.2 + -0.6)$

$= 1.3 + -0.8$

$= 0.5$

[A 13]

Simplify:

$-19 - 3 + 4 - 6 - 17 - 4 + 3 + 19$

Answer on your work sheet.

Turn to panel [A 14] on PAGE 40.

[Q 14]

$(-19 + 19) + (-3 + 3) + (4 + -4) - 6 - 17$

$= -23$

Record your result.

[A 14]

Turn to PAGE 42.

[Part 114] [Page 41]

Check your answers. [A 10]

1. $(5\frac{1}{2} + 9) + (-2\frac{1}{2} + -7\frac{1}{2})$
   $= 14\frac{1}{2} + -10$
   $= 4\frac{1}{2}$

2. $(6\frac{1}{3} + 3\frac{1}{6}) + (-2\frac{1}{4} + -5\frac{1}{2} + -2\frac{2}{3})$
   $= 9\frac{1}{2} + -10\frac{5}{12}$
   $= -\frac{11}{12}$

3. $(4.83 + 7.9) + (-10 + -3.8)$
   $= 12.73 + -13.8$
   $= -1.07$

4. $(16.75 + 40.72) + (-11.3 + -61.007)$
   $= 57.47 + -72.307$
   $= -14.837$

Record your results.

If you simplified all four exercises correctly, turn to PAGE 34.

If you missed one or more of the exercises, turn to [Q 11], PAGE 39.

Tell whether each of the following expressions stands for a positive number or a negative number.

-5

Answer on your work sheet.

[Q 1]

$-2 + 4$

positive

Record your result.

[A 4]

$-(2 + 4)$

Answer on your work sheet.

[Q 5]

-2×4

negative

Record your result.

[A 8]

2×-4

Answer on your work sheet.

[Q 9]

$3 - 8$

negative

Record your result.

[A 12]

$-(3 + 8)$

Answer on your work sheet.

[Q 13]

$(6 - 8) \times (4 - 11)$

positive

Record your result.

[A 16]

$(9 - 5) \times (8 - 11) \times (12 - 2)$

Answer on your work sheet.

[Q 17]

PART 114.5

Part 112 gave meaning to:

subtracting x

<u>for each real number x</u>. The development used to do this is closely paralleled in Part 114.5 where we seek to give meaning to:

dividing by x

<u>for each nonzero real number x</u>. Continuing the pump-movie-film situation introduced in Part 104, we present a physical situation from which we will extract the operation RECIPROCATING:

[Part 114.5] [Page 1]

The inspector at the Zabranchburg Waterworks decides that the water pumps should be checked to see that they are operating at the rates at which they are supposed to operate. Being too lazy to visit each station personally, he requests that the men in charge of the various pumps take motion pictures of the tanks while they are being filled or emptied. Since all of the tanks in Zabranchburg are made of glass, the inspector will be able to tell by looking at each movie just what is happening in the tank for that movie.

The check will be made in the following manner. The films received from the pumping stations will be run so that what is seen on the screen is an <u>increase</u> of <u>1 gallon</u>. Since the inspector knows the rate at which each of the pumps is supposed to operate, he can predict how long each movie should run in order to observe this increase of 1 gallon. By keeping track of the time, he can tell if the pump is operating correctly.

[Part 114.5] [Page 2]

Let's suppose that the inspector is now ready to check the pump at Station A. He can tell by his records that this pump is an EMPTYING pump which is supposed to operate at the rate of 0.5 gallons per minute. In order for the inspector to observe on the screen an INCREASE of 1 gallon in the water-volume of the Station A tank, should he set the projector forward or backward?

Circle the answer on your work sheet.

In the exercises on page 11, the students are asked to find the second number of each of six ordered pairs. This is preparatory to naming

$$\{(x,\ y) :\quad xy = {}^{+}1\} :$$

RECIPROCATING.

[Part 114.5] [Page 6].

Check your answer.

PUMP	RATE	EXPECTED TIME	SCREEN-CHANGE
A	$^{-}0.5$	$^{-}2$	$^{+}1$
B	$^{+}0.1$	$^{+}10$	$^{+}1$

Since the Station B pump is a FILLING pump [positive rate], the movie should be run FORWARD [positive time] in order to observe on the screen an INCREASE in the water-volume.

Record your result.

* * *

Complete the inspector's table for the pumps at Stations C and D. Write the answers on your work sheet. [Use decimals when appropriate.]

	PUMP	RATE	EXPECTED TIME	SCREEN-CHANGE
(1)	C	$^{-}0.25$	?	$^{+}1$
(2)	D	$^{+}2$	?	$^{+}1$

Tables like those on page 6 are designed to lead the student to discover that for a given pump, having a nonzero rate, "the expected time is the reciprocal of the rate of the pump."

[Part 114.5] [Page 11]

Check your answers.

	PUMP	RATE	EXPECTED TIME	SCREEN-CHANGE
	G	$^{+}\frac{1}{7}$	$^{+}7$	$^{+}1$
(1)	H	$^{+}8$	$^{+}\frac{1}{8}$	$^{+}1$
(2)	I	$^{-}3$	$^{-}\frac{1}{3}$	$^{+}1$
(3)	J	$^{-}\frac{1}{3}$	$^{-}3$	$^{+}1$

Record your results.

* * *

In each of the following exercises, you are given a first real number. Consider this as the rate [in gallons per minute] at which a pump works. Then, give a second real number which measures the expected time [in minutes] in order to observe on the screen <u>an increase of 1 gallon</u> in the water-volume.

(1) ($^{+}6$, ?) (2) ($^{-}5$, ?)

(3) ($^{+}\frac{1}{6}$, ?) (4) ($^{+}9$, ?)

(5) ($^{-}15$, ?) (6) ($^{+}2$, ?)

[Part 114.5] [Page 12]

Check your answers.

(1) $(^{+}6, \frac{^{+}1}{6})$ (2) $(^{-}5, \frac{^{-}1}{5})$

(3) $(\frac{^{+}1}{6}, {^{+}6})$ (4) $(^{+}9, \frac{^{+}1}{9})$

(5) $(^{-}15, \frac{^{-}1}{15})$ (6) $(^{+}2, \frac{^{+}1}{2})$

Record your results.

* * *

In the problems you have been working, you have been concerned with pairs of numbers whose product is $^{+}1$. For example:

$$^{+}6 \times \frac{^{+}1}{6} = {^{+}1}$$

$$^{-}5 \times \frac{^{-}1}{5} = {^{+}1}$$

$$^{+}9 \times \frac{^{+}1}{9} = {^{+}1}$$

$$\frac{^{+}1}{6} \times {^{+}6} = {^{+}1}$$

When you start with a first real number, and find a second real number such that their product is $^{+}1$, you are performing a <u>singulary operation</u>. This singulary operation is called:

RECIPROCATING [pronounced: REE·SIP′·RO·KAY·TING]

$(^{-}7, \frac{^{-}1}{7})$ $(\frac{^{+}1}{16}, {^{+}16})$

$(^{-}2, ?)$ $(\frac{^{+}1}{2}, ?)$ $(\frac{^{-}1}{10}, ?)$

$(\frac{^{+}4}{3}, \frac{^{+}3}{4})$ $(\frac{^{-}3}{4}, ?)$

$(^{+}121, ?)$

. . .

Answer the questions on your work sheet.

A worded problem and a bit of machine notation make it clear that zero has no reciprocal.

[Part 114.5] [Page 15]

Check your answers.

(1) $^{+}19$ → reciprocating → $\frac{^{+}1}{19}$

[Check: $^{+}19 \times \frac{^{+}1}{19} = {^{+}1}$]

(2) $\frac{^{+}1}{25}$ → reciprocating → $^{+}25$

[Check: $\frac{^{+}1}{25} \times {^{+}25} = {^{+}1}$]

(3) $^{-}25$ → rec procating → $\frac{^{-}1}{25}$

[Check: $^{-}25 \times \frac{^{-}1}{25} = {^{+}1}$]

Record your results.

* * *

The pump at Station K of the Zabranchburg Water-works broke down and STOPPED just before the attendent took the motion picture of the tank. The film was sent, as usual, to the waterworks inspector. The pump at Station K was a FILLING pump which pumps at the rate of 12 gallons per minute [<u>when it is working</u>]. The inspector's chart for Station K looks like this:

PUMP	RATE	EXPECTED TIME	SCREEN-CHANGE
K	$^{+}12$	$\frac{^{+}1}{12}$	$^{+}1$

After the project is started, how long must the inspector watch the movie of the tank at Station K to observe a change of 1 gallon in the water-volume?

Write your answer on your work sheet.

[Part 114.5] [Page 16]

Check your answer.

If you said that he will never see a change of 1 gallon in the water-volume of the tank at Station K, you are correct.

Remember that because the pump had stopped, it was pumping at the rate of 0 gallons per minute. So, the water-volume in the tank will not change no matter how long the movie is.

ecord your result.

* * *

Suppose we put 0 into the RECIPROCATING machine.

0 → reciprocating- → REJECT

<u>Check</u>: (a real number) × (the reciprocal of the real number) = $^{+}1$

Here is one reason why the RECIPROCATING machine rejects the possibility of 0 having a reciprocal. If there were a real number which is the reciprocal of 0 then

$$\left.\begin{aligned} ^{+}1 &= 0 \times (\text{the reciprocal of } 0) \\ &= (\text{the reciprocal of } 0) \times 0 \end{aligned}\right\} \text{[cpm]}$$

$$\left.\begin{aligned} &= 0. \end{aligned}\right\} \text{[pm0]}$$

In other words, if there were a real number which is the reciprocal of 0, we would have to accept the conclusion that $^{+}1 = 0$. But, $^{+}1 \neq 0$. So, there is no real number which is the reciprocal of 0.

The RECIPROCATING machine rejects 0. For this reason we say that <u>the reciprocating machine works only with nonzero real numbers</u>.

The function notation introduced on page 17 makes it easy for the student to discover and give a pattern sentence for THE PRINCIPLE OF RECIPROCALS.

[Part 114.5] [Page 17]

Instead of writing, say:

the reciprocal of $^{-}8$

Let's agree to write the shorter symbol:

$\mathcal{R}(^{-}8)$ · [Read this as the 'reciprocal of negative eight'.]

So, for example, we can write:

$\mathcal{R}(^{+}5) = {}^{+}\frac{1}{5}$, $\mathcal{R}(^{-}6) = {}^{-}\frac{1}{6}$, and: $\mathcal{R}(^{+}\frac{1}{25}) = {}^{+}25$

Compute. [Write the answers on your work sheet.]

(1) $\mathcal{R}(^{+}\frac{1}{2})$

(2) $\mathcal{R}(^{+}\frac{1}{2}) + \mathcal{R}(^{-}\frac{1}{3})$

(3) $\mathcal{R}(^{+}\frac{1}{6}) + \mathcal{R}(^{-}\frac{1}{6})$

(4) $\mathcal{R}(^{+}6) + \mathcal{R}(^{-}6)$

[Part 114.5] [Page 20]

Check your answers.

(1) $^{+}11 \times \mathcal{R}(^{+}11) = {}^{+}1$

(2) $^{+}21 \times \mathcal{R}(^{+}21) = {}^{+}1$

(3) $^{-}67 \times \mathcal{R}(^{-}67) = {}^{+}1$

(4) $^{-}\frac{1}{24} \times \mathcal{R}(^{-}\frac{1}{24}) = {}^{+}1$

(5) $^{+}957 \times \mathcal{R}(^{+}957) = {}^{+}1$

(6) $^{+}\frac{1}{957} \times \mathcal{R}(^{+}\frac{1}{957}) = {}^{+}1$

Record your results.

* * *

Recall that when the RECIPROCATING machine operates to find the reciprocal of a nonzero real number given to it, the machine follows the principle that

(a nonzero real number) × (the reciprocal of the real number) = $^{+}1$.

Using our new notation, we can write this sentence as follows:

(a nonzero real number) × $\mathcal{R}$(the real number) = $^{+}1$

This principle that the reciprocating machine follows is called:

THE PRINCIPLE OF RECIPROCALS

Pattern: ___ × $\mathcal{R}$(__) = $^{+}1$
nonzero

Complete the following sentences on your work sheet.

(1) An instance of the principle of reciprocals is:
$^{-}12 \times \mathcal{R}(\ ?\ ) = {}^{+}1$

(2) The sentence "$^{+}628 \times \mathcal{R}(^{+}628) = {}^{+}1$" is an instance of ___?___ ___?___ ___?___ ___?___.

Lists are used to help the students arrive at the conclusion that, for each nonzero real number, the inverse of multiplying by that number is the same thing as multiplying by the reciprocal of that number.

Check your answers.

$(^{+}597 \times {}^{-}783) \times \mathcal{R}({}^{-}783) = {}^{+}597 \times [{}^{-}783 \times \mathcal{R}({}^{-}783)]$ [assoc.]

$= {}^{+}597 \times {}^{+}1$ [op.]

$= {}^{+}597$ [op.]

So, $({}^{+}597 \times {}^{-}783) \times \mathcal{R}({}^{-}783) = {}^{+}597$.

Record your results.

* * *

Here is a list of seven pairs for the singulary operation MULTIPLYING BY $^{-}5$.

multiplying by $^{-}5$

$(^{-}17, {}^{+}85)$ $(^{+}30, {}^{-}150)$
$(0, 0)$ $(^{-}5, {}^{+}25)$ $(^{+}1, {}^{-}5)$
$(\frac{^{-}15}{2}, \frac{^{+}75}{2})$ $(\frac{^{+}23}{5}, {}^{-}23)$
...

This singulary operation has an INVERSE -- that is, there is a singulary operation which undoes what multiplying by $^{-}5$ did.

On your work sheet give seven pairs which belong to THE INVERSE OF MULTIPLYING by $^{-}5$ and which correspond to those given above.

Check your answers.

multiplying by $^{-}5$	the inverse of multiplying by $^{-}5$
$(^{-}17, {}^{+}85)$	$(^{+}85, {}^{-}17)$
$(^{+}30, {}^{-}150)$	$(^{-}150, {}^{+}30)$
$(0, 0)$	$(0, 0)$
$(^{-}5, {}^{+}25)$	$(^{+}25, {}^{-}5)$
$(^{+}1, {}^{-}5)$	$(^{-}5, {}^{+}1)$
$(\frac{^{-}15}{2}, \frac{^{+}75}{2})$	$(\frac{^{+}75}{2}, \frac{^{-}15}{2})$
$(\frac{^{+}23}{5}, {}^{-}23)$	$(^{-}23, \frac{^{+}23}{5})$
...	...

Record your results.

* * *

Find the missing numbers for this list of seven pairs for the singulary operation MULTIPLYING BY $\mathcal{R}(^{-}5)$.

multiplying by $\mathcal{R}(^{-}5)$

$(^{+}85, {}^{-}17)$ $(^{-}150, \ ?\ )$
$(0, \ ?\ )$ $(^{+}25, \ ?\ )$ $(^{-}5, \ ?\ )$
$(\frac{^{+}75}{2}, \ ?\ )$ $(^{-}23, \ ?\ )$
...

Check your answers.

multiplying by $\mathcal{R}(^{-}5)$

$(^{+}85, {}^{-}17)$ $(^{-}150, {}^{+}30)$ (a)
(b) $(0, 0)$ $(^{+}25, {}^{-}5)$ $(^{-}5, {}^{+}1)$ (d)
(c) $(\frac{^{+}75}{2}, \frac{^{-}15}{2})$ $(^{-}23, \frac{^{+}23}{5})$ (f)
(e)
...

Record your results.

* * *

Find the missing numbers for these lists.

multiplying by $^{+}10$	the inverse of multiplying by $^{+}10$	multiplying by $\mathcal{R}(^{+}10)$
$(^{+}7, {}^{+}70)$	$(^{+}70, \ ?\ )$	$(^{+}70, \ ?\ )$
$(^{-}2, \ ?\ )$	$(^{-}20, \ ?\ )$	$(^{-}20, \ ?\ )$
$(^{+}1, \ ?\ )$	$(^{+}10, \ ?\ )$	$(^{+}10, \ ?\ )$
$(^{-}5, \ ?\ )$	$(^{-}50, \ ?\ )$	$(^{-}50, \ ?\ )$
$(^{-}3, \ ?\ )$	$(^{-}30, \ ?\ )$	$(^{-}30, \ ?\ )$
$(0, \ ?\ )$	$(0, \ ?\ )$	$(0, \ ?\ )$
...	...	...

Check your answers.

multiplying by $^{+}10$	the inverse of multiplying by $^{+}10$	multiplying by $\mathcal{R}(^{+}10)$
$(^{+}7, {}^{+}70)$	$(^{+}70, {}^{+}7)$	$(^{+}70, {}^{+}7)$
$(^{-}2, {}^{-}20)$	$(^{-}20, {}^{-}2)$	$(^{-}20, {}^{-}2)$
$(^{+}1, {}^{+}10)$	$(^{+}10, {}^{+}1)$	$(^{+}10, {}^{+}1)$
$(^{-}5, {}^{-}50)$	$(^{-}50, {}^{-}5)$	$(^{-}50, {}^{-}5)$
$(^{-}3, {}^{-}30)$	$(^{-}30, {}^{-}3)$	$(^{-}30, {}^{-}3)$
$(0, 0)$	$(0, 0)$	$(0, 0)$
...	...	...

Record your results.

* * *

Answer these questions on your work sheet.

(1) The singulary operation MULTIPLYING BY $^{+}10$ takes you, for example, from $^{+}7$ to __?__.

(2) THE INVERSE OF MULTIPLYING BY $^{+}10$ is a singulary operation which undoes what MULTIPLYING BY $^{+}10$ does. So, the inverse of multiplying by $^{+}10$ takes you from $^{+}70$ to __?__.

(3) The singulary operation MULTIPLYING BY $\mathcal{R}(^{+}10)$ also takes you from $^{+}70$ to __?__.

Our next move is to agree that, for each nonzero real number, the inverse of multiplying by that number is the same as dividing by that number. Having accomplished this on page 36, the tables on page 38 allow us to define (on page 40) what we mean by: dividing by a nonzero real number x

[Part 114.5] [Page 36]

Check your answers.

(1) $^-3 \times {}^+3 = {}^-9$ and $^-9 \times \underbrace{\mathcal{R}(^+3)}_{\text{apply the inverse of multiplying by } {}^+3} = {}^-9 \times {}^+\frac{1}{3} = {}^-3$

(2) $^-13 \times {}^-5 = {}^+65$ and $^+65 \times \underbrace{\mathcal{R}(^-5)}_{\text{apply the inverse of multiplying by } {}^-5} = {}^+65 \times {}^-\frac{1}{5} = {}^-13$

(3) $^+5 \times {}^+22 = {}^+110$ and $^+110 \times \mathcal{R}(^+22) = {}^+110 \times {}^+\frac{1}{22} = {}^+5$

Record your results.

* * *

In an earlier part of the course you learned that for each number of arithmetic:

DIVIDING BY THAT NUMBER

meant nothing more than what is meant by the phrase:

THE INVERSE OF MULTIPLYING BY THAT NUMBER

Similarly, for example, when we say:

THE INVERSE OF MULTIPLYING BY $^+3$,

we could more simply say:

DIVIDING BY $^+3$

Answer these questions on your work sheet.

(1) A simpler way of saying: THE INVERSE OF MULTIPLYING BY $^-5$ is to say: ___?___ ___?___ ___?___

(2) The phrase 'DIVIDING BY $^+22$' means precisely the same thing as the phrase '___?___ ___?___ ___?___ ___?___ ___?___ ___?___'.

[Part 114.5] [Page 37]

Check your answers.

(1) A simpler way of saying: THE INVERSE OF MULTIPLYING BY $^-5$ is to say: DIVIDING BY $^-5$

(2) The phrase 'DIVIDING BY $^+22$' means precisely the same thing as the phrase 'THE INVERSE OF MULTIPLYING BY $^+22$'.

Record your results.

* * *

Let's agree that for each nonzero real number:

dividing by that number

means the same thing as:

the inverse of multiplying by that number

Hence, any problem in which you divide by a nonzero real number is equivalent to a problem in applying the inverse of multiplying by that nonzero real number.

Answer these questions on your work sheet.

(1)

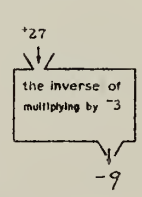
(2)

(3)

[Part 114.5] [Page 38]

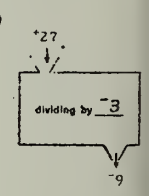
Check your answers.

(1)



(2)

(3)



Record your results.

* * *

Fill the blanks on your work sheet.

Number	Pair belonging to MULTIPLYING BY $\mathcal{R}(^-3)$	Corresponding pair belonging to THE INVERSE of MULTIPLYING BY $^-3$	Corresponding pair belonging to DIVIDING BY $^-3$
(a) $^-6$	$(^-6, {}^+2)$	$(^-6, {}^+2)$	$(^-6, {}^+2)$
(b) $^-1$	$(^-1, {}^+\frac{1}{3})$	$(^-1, {}^+\frac{1}{3})$	$(^-1, {}^+\frac{1}{3})$
(1) $^+6$	$(^+6, ?)$	$(?, ?)$	$(^+6, ?)$
(2) $^-15$	$(?, ?)$	$(^-15, ?)$	$(^-15, ?)$
(3) $^+\frac{1}{3}$	$(^+\frac{1}{3}, ?)$	$(?, ?)$	$(^+\frac{1}{3}, ?)$
(4) $^-20$	$(^-20, ?)$	$(^-20, ?)$	$(^-20, ?)$
(5) $^+99$	$(?, ?)$	$(^+99, ?)$	$(^+99, ?)$

[Part 114.5] [Page 39]

Check your answers.

Number	Pair belonging to MULTIPLYING BY $\mathcal{R}(^-3)$	Corresponding pair belonging to THE INVERSE of MULTIPLYING BY $^-3$	Corresponding pair belonging to DIVIDING BY $^-3$
(1) $^+6$	$(^+6, ^-2)$	$(^+6, ^-2)$	$(^+6, ^-2)$
(2) $^-15$	$(^-15, ^+5)$	$(^-15, ^+5)$	$(^-15, ^+5)$
(3) $^+\frac{1}{3}$	$(^+\frac{1}{3}, ^-\frac{1}{9})$	$(^+\frac{1}{3}, ^-\frac{1}{9})$	$(^+\frac{1}{3}, ^-\frac{1}{9})$
(4) $^-20$	$(^-20, ^+\frac{20}{3})$	$(^-20, ^+\frac{20}{3})$	$(^-20, ^+\frac{20}{3})$
(5) $^+99$	$(^+99, ^-33)$	$(^+99, ^-33)$	$(^+99, ^-33)$

Record your results.

* * *

as you discovered earlier, a problem in applying the singulary operation MULTIPLYING BY THE RECIPROCAL OF A NONZERO REAL NUMBER is nothing more than a problem in applying THE INVERSE OF MULTIPLYING BY THAT REAL NUMBER.

By the agreement made on Page 37, we know that a problem in applying THE INVERSE OF MULTIPLYING BY A NONZERO REAL NUMBER is equivalent to a problem in applying the singulary operation DIVIDING BY THAT REAL NUMBER.

So, we can conclude that, no matter what nonzero real number we might pick,

MULTIPLYING BY ? ? ? ? ?
is the same thing as
? BY THAT NUMBER

Fill the blanks on your work sheet.

[Part 114.5] [Page 40]

Check your answers.

For each nonzero real number,
MULTIPLYING BY THE RECIPROCAL OF THAT NUMBER
is the same thing as
DIVIDING BY THAT NUMBER.

Record your results.

* * *

The principle for real numbers stated in the box above tells us, for example, that

(a) $^-45 \times \mathcal{R}(^-9) = {}^-45 \div {}^-9$ [or: $\frac{^-45}{^-9}$]

(b) $^-45 \times \frac{^-1}{9} = {}^-45 \div {}^-9$ [or: $\frac{^-45}{^-9}$]

(c) $^+28 \times \frac{^+1}{7} = \frac{^+28}{^+7}$ [that is: $^+28 \times \frac{^+1}{7} = {}^+28 \times \mathcal{R}(^+7) = \frac{^+28}{^+7}$]

Use the real number principle stated above to change each of the following multiplication problems into an equivalent division problem.

(1) $^+63 \times \mathcal{R}(^-7) \rightarrow ? \div ?$

(2) $^-56 \times \frac{^+1}{4} \rightarrow \frac{?}{?}$

(3) $^+937 \times \frac{^+1}{81} \rightarrow \frac{?}{?}$

Tables like those on pages 47-49 help students to discover theorems.

[Part 114.5] [Page 47]

Check your answers.

(1) $^-5 \div \frac{^-1}{5} = {}^-5 \times {}^-5 = {}^+25$

(2) $^-5 \div \mathcal{R}(^-5) = {}^-5 \div \frac{^-1}{5} = {}^+25$

(3) $^-5 \div \mathcal{O}(\frac{1}{5}) = {}^-5 \div \frac{^-1}{5} = {}^-5 \div \frac{^-1}{5} = {}^+25$

(4) $^-5 \div \mathcal{O}(\mathcal{R}(5)) = {}^-5 \div \mathcal{O}(\frac{1}{5}) = {}^-5 \div \frac{^-1}{5} = {}^+25$

Record your results.

* * *

Fill the blanks on your work sheet.

	a number	$\mathcal{O}$(the number)	$\mathcal{R}$(the number)	$\mathcal{O}(\mathcal{R}$(the number))	$\mathcal{R}(\mathcal{O}$(the number))
(a)	$^+3$	$^-3$	$\frac{1}{3}$	$\mathcal{O}(\frac{1}{3}) = \frac{^-1}{3}$	$\mathcal{R}(^-3) = \frac{^-1}{3}$
(1)	$^+5$	?	?	?	?
(2)	$^-8$	?	?	?	?
(3)	$^+\frac{2}{3}$	?	?	?	?
(4)	$^+\frac{7}{2}$	?	?	?	?
(5)	$\frac{^-121}{13}$	?	?	?	?

[Part 114.5] [Page 48]

Check your answers.

	a number	$\mathcal{O}$(the number)	$\mathcal{R}$(the number)	$\mathcal{O}(\mathcal{R}$(the number))	$\mathcal{R}(\mathcal{O}$(the number))
(1)	$^{+}5$	$^{-}5$	$^{+}\frac{1}{5}$	$^{-}\frac{1}{5}$	$^{-}\frac{1}{5}$
(2)	$^{-}8$	$^{+}8$	$^{-}\frac{1}{8}$	$^{+}\frac{1}{8}$	$^{+}\frac{1}{8}$
(3)	$^{+}\frac{2}{3}$	$^{-}\frac{2}{3}$	$^{+}\frac{3}{2}$	$^{-}\frac{3}{2}$	$^{-}\frac{3}{2}$
(4)	$^{+}\frac{7}{2}$	$^{-}\frac{7}{2}$	$^{+}\frac{2}{7}$	$^{-}\frac{2}{7}$	$^{-}\frac{2}{7}$
(5)	$^{-}\frac{121}{13}$	$^{+}\frac{121}{13}$	$^{-}\frac{13}{121}$	$^{+}\frac{13}{121}$	$^{+}\frac{13}{121}$

Record your results.

* * *

Fill the blanks on your work sheet.

	a number	$\mathcal{O}$(the number)	$\mathcal{R}$(the number)	$\mathcal{O}(\mathcal{R}$(the number))	$\mathcal{R}(\mathcal{O}$(the number))
(1)	$^{-}\frac{1}{2}$	?	?	?	?
(2)	$^{+}\frac{1}{2}$	?	?	?	?
(3)	$^{+}1000$	?	?	?	?
(4)	0	?	?	?	?

[Part 114.5] [Page 49]

Check your answers.

	a number	$\mathcal{O}$(the number)	$\mathcal{R}$(the number)	$\mathcal{O}(\mathcal{R}$(the number))	$\mathcal{R}(\mathcal{O}$(the number))
(1)	$^{-}\frac{1}{2}$	$^{+}\frac{1}{2}$	$^{-}2$	$^{+}2$	$^{+}2$
(2)	$^{+}\frac{1}{2}$	$^{-}\frac{1}{2}$	$^{+}2$	$^{-}2$	$^{-}2$
(3)	$^{+}1000$	$^{-}1000$	$^{+}\frac{1}{1000}$	$^{-}\frac{1}{1000}$	$^{-}\frac{1}{1000}$
(4)	0	0	nonsense	nonsense	nonsense

Record your results.

* * *

The results you obtained above seem to indicate that:

For each ___?___ real number

THE OPPOSITE OF THE ___?___ OF THAT NUMBER

is the same as

THE RECIPROCAL OF THE ___?___ OF THAT NUMBER.

Fill the blanks on your work sheet.

Exercises on page 70 suggest the existence of

THE DISTRIBUTIVE PRINCIPLE FOR DIVISION OVER ADDITION.

This provides an opportunity to use

THE PRINCIPLE OF RECIPROCALS

in a proof.

[Part 114.5] [Page 70]

Check your answers.

(1) $^{+}4 \div {}^{+}\frac{1}{2} = {}^{+}8$ $\quad [{}^{+}8 \times {}^{+}\frac{1}{2} = {}^{+}4\ ]$

(2) $(^{+}7 \times {}^{+}4) \div {}^{+}\frac{1}{2} = {}^{+}8$ $\quad [{}^{+}8 \times {}^{+}\frac{1}{2} = {}^{+}4 = ({}^{+}1 \times {}^{+}4)]$

(3) $[(^{-}\frac{1}{2} \times {}^{-}2) \times {}^{+}4] \div {}^{+}\frac{1}{2} = {}^{+}8$

Record your results.

* * *

Do these problems.

(1) $(^{+}6 \div {}^{-}2) + (^{+}8 \div {}^{-}2) =$ ___?___ $\div\ ^{-}2 =$ ___?___

(2) $(^{-}5 \div {}^{-}4) + (^{+}7 \div {}^{-}4) =$ ___?___ $\div\ ^{-}4 =$ ___?___

(3) $(^{-}2 \div {}^{+}3) + (^{+}14 \div {}^{+}3) = {}^{+}12 \div$ ___?___ $=$ ___?___

(4) $(^{-}21 \div {}^{+}8) + (^{-}3 \div {}^{+}8) = {}^{-}24 \div$ ___?___ $=$ ___?___

[Part 114.5] [Page 71]

Check your answers.

(1) $(^{+}6 \div {}^{-}2) + (^{+}8 \div {}^{-}2) = {}^{+}14 \div {}^{-}2 = {}^{-}7$

(2) $({}^{-}5 \div {}^{-}4) + ({}^{+}7 \div {}^{-}4) = {}^{+}2 \div {}^{-}4 = {}^{-}\frac{1}{2}$

(3) $({}^{+}2 \div {}^{+}3) + ({}^{+}14 \div {}^{+}3) = {}^{+}12 \div {}^{+}3 = {}^{+}4$

(4) $({}^{-}21 \div {}^{+}8) + ({}^{-}3 \div {}^{+}8) = {}^{-}24 \div {}^{+}8 = {}^{-}3$

Record your results.

* * *

From your work in an earlier part of this course you learned, for example, that

(☆) $({}^{+}6 \times {}^{-}2) + ({}^{+}8 \times {}^{-}2) = ({}^{+}6 + {}^{+}8) \times {}^{-}2$

and that

(*) $({}^{-}5 \times {}^{-}4) + ({}^{+}7 \times {}^{-}4) = ({}^{-}5 + {}^{+}7) \times {}^{-}4.$

Each of sentences (☆) and (*) is an instance of

THE DISTRIBUTIVE PRINCIPLE FOR MULTIPLICATION OVER ADDITION.

From example (1), above, we see that:

$({}^{+}6 \div {}^{-}2) + ({}^{+}8 \div {}^{-}2) = ({}^{+}6 + {}^{+}8) \div {}^{-}2$

From example (2), above, we see that:

$({}^{-}5 \div {}^{-}4) + ({}^{+}7 \div {}^{-}4) = ({}^{-}5 + {}^{+}7) \div {}^{-}4$

The problems above are instances of:

THE ___?___ PRINCIPLE FOR ___?___ OVER ADDITION.

[Part 114.5] [Page 72]

Check your answers.

The sentences:

$({}^{+}6 \div {}^{-}2) + ({}^{+}8 \div {}^{-}2) = ({}^{+}6 + {}^{+}8) \div {}^{-}2$

and: $({}^{-}5 \div {}^{-}4) + ({}^{+}7 \div {}^{-}4) = ({}^{-}5 + {}^{+}7) \div {}^{-}4$

are instances of

THE *DISTRIBUTIVE* PRINCIPLE FOR *DIVISION* OVER ADDITION.

Record your results.

* * *

Consider the sentence:

$({}^{-}759 \div {}^{+}29) + ({}^{+}863 \div {}^{+}29) = ({}^{-}759 + {}^{+}863) \div {}^{+}29$

This sentence is an instance of the distributive principle for division over addition. Without doing any computing, we can show that this sentence is correct. All we have to do is to show that it is a consequence of other principles which we already accept. [Fill the blanks on your work sheet.]

$({}^{-}759 \div {}^{+}29) + ({}^{+}863 \div {}^{+}29)$

$= [\ ?\ \times \mathcal{R}({}^{+}29)] + [{}^{+}863 \times \mathcal{R}({}^{+}29)]$

$= ({}^{-}759 + \ ?\ ) \times \mathcal{R}(\ ?\ )$

$= ({}^{-}759 + {}^{+}863) \div\ ?$

{ dividing by $^{+}29$ is the same as multiplying by the ___?___ of $^{+}29$ [dpma]

{ multiplying by $\mathcal{R}$ ($^{+}29$) is the same as ___?___ by $^{+}29$

So, $({}^{-}759 \div {}^{+}29) + ({}^{+}863 \div {}^{+}29) = ({}^{-}759 + {}^{+}863) \div\ ?$

[Part 114.5] [Page 73]

Check your answers.

$({}^{-}759 \div {}^{+}29) + ({}^{+}863 \div {}^{+}29)$

$= [{}^{-}759 \times \mathcal{R}({}^{+}29)] + [{}^{+}863 \times \mathcal{R}({}^{+}29)]$

$= ({}^{-}759 + {}^{+}863) \times \mathcal{R}({}^{+}29)$

$= ({}^{-}759 + {}^{+}863) \div {}^{+}29$

{ dividing by $^{+}29$ is the same as multiplying by the *reciprocal* of $^{+}29$ [dpma]

{ multiplying by $\mathcal{R}$ ($^{+}29$) is the same as *dividing* by $^{+}29$

So, $({}^{-}759 \div {}^{+}29) + ({}^{+}863 \div {}^{+}29) = ({}^{-}759 + {}^{+}863) \div {}^{+}29.$

Record your results.

* * *

We used the DISTRIBUTIVE PRINCIPLE FOR MULTIPLICATION OVER ADDITION to help us prove the DISTRIBUTIVE PRINCIPLE FOR DIVISION OVER ADDITION. Hence, we can say that

THE DISTRIBUTIVE PRINCIPLE FOR DIVISION OVER ADDITION

is a consequence of

THE DISTRIBUTIVE PRINCIPLE FOR MULTIPLICATION OVER ADDITION.

A pattern-sentence for the DPMA is:

(___ + ⌒)× ..... = (___× .....) + (⌒× .....)

Write a pattern-sentence for THE DISTRIBUTIVE PRINCIPLE FOR DIVISION OVER ADDITION on your work sheet.

The closing pages of Part 114.5 provide another opportunity for the student to employ counterexamples.

[Part 114.5] [Page 74]

Check your answer.

A pattern-sentence for the distributive principle for division over addition is:

$$(___ + \sim) \div \ldots\ldots = (___ \div \ldots\ldots) + (\sim \div \ldots\ldots)$$

Record your result.

* * *

An instance of THE LEFT DISTRIBUTIVE PRINCIPLE FOR MULTIPLICATION OVER ADDITION is

$$^{+}15 \times (^{-}23 + {}^{+}25) = ({}^{+}15 \times {}^{-}23) + ({}^{+}15 \times {}^{+}25)$$

A pattern-sentence for LDPMA is

$$\ldots\ldots \times (___ + \sim) = (\ldots\ldots \times ___) + (\ldots\ldots \times \sim)$$

Are you willing to accept a corresponding left distributive principle for division over addition? [Circle 'yes' or 'no' on your work sheet.]

[Part 114.5] [Page 75]

Check your answer.

If you circled *NO*, you are correct. It is enough to show one instance which is a false statement in order to prove that there is no left distributive principle for division over addition.

Here is one such example:

$$^{-}12 \div ({}^{+}1 + {}^{-}3) = ({}^{-}12 \div {}^{+}1) + ({}^{-}12 \div {}^{-}3)$$

This sentence is <u>false</u> because:

$^{-}12 \div ({}^{+}1 + {}^{-}3)$	$^{-}12 \div {}^{+}1 + {}^{-}12 \div {}^{-}3$
$= {}^{-}12 \div {}^{-}2$	$= {}^{-}12 + {}^{+}4$
$= {}^{+}6$	$= {}^{-}8$

Record your result.

* * *

This is the end of Part 114.5. Put your work sheet under the front cover of this booklet and return it to your teacher.

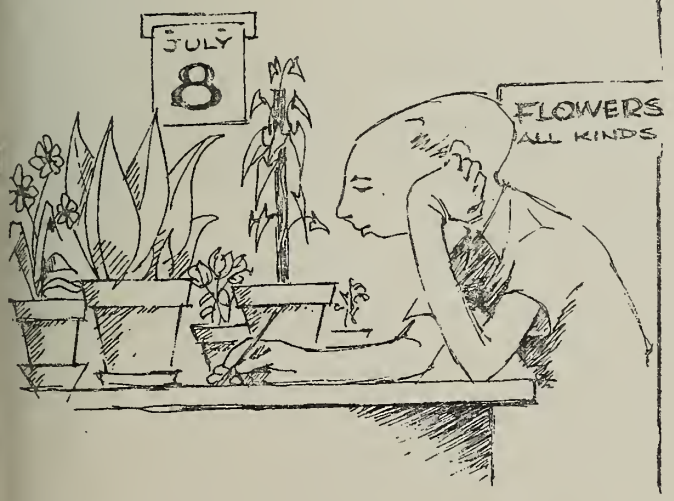
PART 115

On the first five pages of Part 115, we present a situation through which we hope to motivate the students to define the greater-than relation for the real numbers.

[Part 115] [Page 1]

Last summer Bill had a flower business. He kept a record of his daily sales and expenses. Here are two entries from his record.

	Expenses	Sales
July 7	$12	$16
July 8	$15	$17



Which was the "better" day for Bill, July 7 or July 8?

Write the date of the better day on your work sheet.

[Part 115] [Page 2]

Check your answer.

	Expenses	Sales	Outcome
July 7	$12	$16	profit of $4
July 8	$15	$17	profit of $2

Most people would agree that the way to tell is to look at the outcome. July 7 was the better day for Bill's business. On the 7th, Bill made a profit of $4. On the 8th, his profit was only $2.

Record your result.

* * *

Which of these was the better day for Bill?

	Expenses	Sales
July 11	$13	$15
July 12	$16	$14

Write the date of the better day on your work sheet.

[Part 115] [Page 3]

Check your answer.

	Expenses	Sales	Outcome
July 11	$13	$15	profit of $2
July 12	$16	$14	loss of $2

On July 11, Bill made a $2 profit. On July 12 he had a $2 loss. Most people would agree that July 11 was the better day.

Record your result.

* * *

Which of these was the better day for Bill?

	Expenses	Sales
August 3	$17	$12
August 8	$13	$11

Write the date of the better day on your work sheet.

[Part 115] [Page 4]

Check your answers.

	Expenses	Sales	Outcome
August 3	$17	$12	loss of $5
August 8	$13	$11	loss of $2

Although neither of these days was a good one for Bill, he would certainly say that business on August 8 was better than it was on August 3. On August 8 he lost $2, but on August 3 he lost $5. If you must take a loss, it is better to lose $2 than $5.

Record your results.

* * *

Which of these was the better day for Bill?

	Expenses	Sales
July 13	$10	$13
July 17	$12	$ 7

Write the date of the better day on your work sheet.

[Part 115] [Page 5]

Check your answers.

	Expenses	Sales	Outcome
July 13	$10	$13	profit of $3
July 17	$12	$ 7	loss of $5

July 13 was the better day for Bill. A profit is better than a loss.

Record your results.

* * *

Most people would agree that

(a) a large profit is better than a small profit;

(b) any profit is better than a loss;

and (c) a small loss is better than a large loss.

For each pair of entries, tell which is the better day.

		Expenses	Sales
(1)	July 1	$11	$17
	July 2	$ 9	$16
(2)	July 10	$12	$10
	July 14	$13	$16
(3)	July 22	$14	$11
	July 29	$16	$12

Write the answers [dates] on your work sheet.

By the time the student reaches page 18 he has been well motivated to accept the use of real numbers in measuring business outcomes.

[Part 115] [Page 18]

Check your answer.

Saying that August 12 was better than August 15 is equivalent to saying that August 15 was a poorer day than August 12.

Record your result.

* * *

In earlier parts, we have shown that daily business outcomes are measured by real numbers. We have usually agreed to measure profits by positive real numbers.

With this agreement, give the real number which measures [in dollars] the following outcomes.

(1) $4 profit (2) $3 loss

(3) $1 profit (4) break even

(5) $177 loss (6) $3.50 profit

Write the answers on your work sheet.

[Part 115] [Page 19]

Check your answers.

	Outcome	Real number measure
(1)	$4 profit	$^{+}4$
(2)	$3 loss	$^{-}3$
(3)	$1 profit	$^{+}1$
(4)	break even	0
(5)	$177 loss	$^{-}177$
(6)	$3.50 profit	$^{+}3.5$

Record your results.

Did you remember to use negative numbers to measure losses and the real number 0 for a break-even outcome?

* * *

Complete this table. Write answers on your work sheet.

		Expenses	Sales	Outcome
	August 1	11	17	$^{+}6$
	August 2	11	11	0
	August 3	17	12	$^{-}5$
(1)	August 4	18	11	?
(2)	August 5	13	18	?
(3)	August 6	13	13	?
(4)	August 7	15	?	$^{+}2$
(5)	August 8	13	11	?
(6)	August 9	?	16	$^{+}4$

Since the student decides which of two days was a better day for the flower business by comparing the real number outcomes of those days, the next natural step is to compare these outcomes by using the "is-a-better-outcome-than" relation on page 21.

[Part 115] [Page 21]

Check your answers.

Here is the correct list.

$^{-}7$. $^{-}2$. 0 . $^{+}2$. $^{+}4$. $^{+}5$

Record your results.

* * *

Consider the real numbers $^{+}4$ and $^{+}1$. $^{+}4$ measures a better outcome than $^{+}1$; so, we might say:

$^{+}4$ is a better outcome than $^{+}1$

Complete the following on your work sheet.

Consider: (1) $^{+}8$ and $^{+}10$? is a better outcome than ? .

(2) $^{-}6$ and $^{-}7$? is a better outcome than ? .

(3) $^{+}4$ and 0 ? is a better outcome than ? .

(4) $^{-}3$ and $^{+}3$? is a better outcome than ? .

(5) $^{-}5$, 0 ? is a better outcome than ? .

Our "is-a-better-outcome-than" way of comparing two real numbers takes on list form on page 25 and is ready to be given the name:

THE GREATER - THAN RELATION

on page 27. The student has long been ready for the '>' symbol to help him reduce the labor of writing comparisons of two real numbers.

The exercises on pages 31 and 32 give the student an opportunity to discover that THE GREATER - THAN RELATION is transitive.

[Part 115] [Page 25]

Check your answers.

The numeral for the better outcome in each pair is circled.

(a) (+4, +1 (b) +8, (+10) (c) (−6, −7

(d) (+4, 0 (e) −3, (+3) (f) −5, (0)

Record your results.

* * *

Here is a list of pairs of first and second real numbers. Think of the real numbers as measures of daily outcomes with the second real number in each pair measuring a better outcome than the first. There is exactly one mistake in the list. Find it. Use an arrow to point it out on the copy of the list shown on your work sheet.

(−5, +1) (−5, 0) (−5, +7)
(+2, +7) (−3, −1) (+1, +3) (0, +6)
(−6, 0) (+8, +12) (+8, +14) (−8, −7)
(+4, +4.1) (+3.9, +4) (−3.1, −3.2) (+3.1, +3.2)

[Part 115] [Page 26]

Check your answer.

(−5, +1) (−5, 0) (−5, +7)
(+2, +7) (−3, −1) (+1, +3) (0, +6)
(−6, 0) (+8, +12) (+8, +14) (−8, −7)
(+4, +4.1) (+3.9, +4) [−3.1, −3.2] (+3.1, +3.2)

This pair is not correct.

Here is the error. −3.1 measures a better outcome than −3.2. [A loss of $3.10 is better than a loss of $3.20.]

Record your result.

* * *

Here is a list of 5 real numbers.

−3.5 −2 0 +1.9 +4

Use these five numbers to make up pairs such that the second number of each pair measures a better outcome than the first. List the pairs in the squares provided on your work sheet. List as many as you can. [One correct example is (+19, +4).]

[Part 115] [Page 27]

Check your answers.

The five numbers you were to use are

−3.5, −2, 0, +1.9, +4 .

The list below contains all possible correct pairs.

(−3.5, −2) (−3.5, 0) (−3.5, +1.9) (−3.5, +4)
(−2, 0) (−2, +1.9) (−2, +4)
(0, +1.9) (0, +4)
(+1.9, +4)

In each pair, the second number measures a better outcome than the first number.

Record your results. Mark yourself correct if you listed at least five pairs, and if all of the pairs you did list are included above.

* * *

The set of all pairs of first and second real numbers for which the second real number measures a better business outcome than the first is an example of a RELATION among real numbers. This particular relation is called THE GREATER-THAN RELATION.

We say, for example, that (+8, +12) belongs to the greater-than relation. That is, we say: +12 is greater than +8

Another way of saying the same thing is to write:

+12 > +8

This is also read as: positive twelve is greater than positive eight [Notice that the '>' is pronounced as 'is greater than'.]

Similarly, (−4, −1) belongs to the greater-than relation.

So, we say that

−1 is greater than ___?___ .

Also, ___?___ > −4 .

Fill the blanks on your work sheet.

[Part 115] [Page 31]

Check your answers.

(a) The sentence: $^{-}8 > {}^{-}11$

is read as: negative eight *is greater than* negative eleven

(b) The sentence: $^{+}3 > {}^{+}2.99$

is read as: *positive three is greater than* positive two and ninety-nine one hundreths

Record your results.

* * *

Anne, Janet, and Beverly each picked a real number and told them to the teacher. The teacher wrote the following sentences on the blackboard.

Anne's number > Janet's number
Janet's number > Beverly's number

On your work sheet, compare Anne's number with Beverly's.

[Part 115] [Page 32]

Check your answers.

Anne's number > Janet's number
Janet's number > Beverly's number

It follows from the above statements that
Anne's number > Beverly's number
Record your results.

* * *

Answer these questions on your work sheet.

Sample. Just by knowing that $^{-}8 > {}^{-}16$ and that $^{-}16 > {}^{-}21$, we can conclude that $^{-}8 > \underline{^{-}21}$.

(1) Just by knowing that $^{+}14 > {}^{-}3$,
and that $^{-}3 > {}^{-}23$,
we can correctly conclude that $^{+}14 \ \underline{?}\ {}^{-}23$.

(2) Just by knowing that $^{+}1.1 > {}^{+}0.9$,
and that $^{+}0.9 > {}^{+}0.8$,
we can correctly conclude that $^{+}1.1 > \underline{?}$.

(3) Just by knowing that $^{-}3 > {}^{-}3.1$,
and that $^{-}3.1 > {}^{-}3.2$,
we can correctly conclude that $\underline{?} > {}^{-}3.2$.

Pages 33-37 comprise the final step in a development which, hopefully, makes our definition of THE GREATER - THAN RELATION (page 37) seem quite reasonable.

[Part 115] [Page 33]

Check your answers.

(1) Just by knowing that $^{+}14 > {}^{-}3$,
and that $^{-}3 > {}^{-}23$,
we know that $^{+}14 > {}^{-}23$.

(2) Just by knowing that $^{+}1.1 > {}^{+}0.9$,
and that $^{+}0.9 > {}^{+}0.8$,
we can correctly conclude that $^{+}1.1 > \underline{^{+}0.8}$.

(3) Just by knowing that $^{-}3 > {}^{-}3.1$,
and that $^{-}3.1 > {}^{-}3.2$,
we can correctly conclude that $\underline{^{-}3} > {}^{-}3.2$.

Record your results.

* * *

By putting a stroke through the greater-than sign, we obtain the sign that is generally used for is-not-greater-than.

For example: $^{+}7 \not> {}^{+}9$
is read as: positive seven is not greater than positive nine

For each pair of first and second real numbers, either the second is greater than the first or the second is not greater than the first.

Complete the following table. Write the answers on your work sheet. [The first two are worked for you.]

(a) $^{+}7 \not> {}^{+}9$ positive seven *is not greater than* positive nine
(b) $^{-}7 > {}^{-}9$ negative seven *is greater than* negative nine

(1) $^{+}6 \not> {}^{+}11$ positive six ___?___ positive eleven
(2) $^{-}3 > {}^{-}4$ negative three ___?___ negative four
(3) $^{+}5$? $^{+}5$ positive five ___?___ positive five
(4) 0 ? $^{-}4$ zero ___?___ negative four
(5) $^{-}4$? 0 negative four ___?___ zero

Check your answers.

(1) $^{+}6 \not> {}^{+}11$ positive six *is not greater than* positive eleven

(2) $^{-}3 > {}^{-}4$ negative three *is greater than* negative four

(3) $^{+}5 \not> {}^{+}5$ positive five *is not greater than* positive five

(4) $0 > {}^{-}4$ zero *is greater than* negative four

(5) $^{-}4 \not> 0$ negative four *is not greater than* zero

Record your results.

* * *

Fill the blanks on your work sheet. [Use only numerals, '>'s, or '$\not>$'s.]

(1) $^{+}9 > {}^{+}7$ and $^{+}9 = {}^{+}7 + \underline{?}$

(2) $^{+}3 > {}^{-}4$ and $^{+}3 = {}^{-}4 + \underline{?}$

(3) $^{-}4 \underline{?} {}^{-}9$ and $^{-}4 = {}^{-}9 + \underline{?}$

(4) $^{+}5 \not> {}^{+}8$ and $^{+}5 = {}^{+}8 + \underline{?}$

(5) $^{-}10 \not> {}^{-}2$ and $^{-}10 = {}^{-}2 + \underline{?}$

(6) $^{+}4 \underline{?} {}^{+}10$ and $^{+}4 = {}^{+}10 + \underline{?}$

(7) $^{+}6 \not> {}^{+}6$ and $^{+}6 = {}^{+}6 + \underline{?}$

Check your answers.

(1) $^{+}9 > {}^{+}7$ and $^{+}9 = {}^{+}7 + \underline{^{+}2}$

(2) $^{+}3 > {}^{-}4$ and $^{+}3 = {}^{-}4 + \underline{^{+}7}$

(3) $^{-}4 > {}^{-}9$ and $^{-}4 = {}^{-}9 + \underline{^{+}5}$

(4) $^{+}5 \not> {}^{+}8$ and $^{+}5 = {}^{+}8 + \underline{^{-}3}$

(5) $^{-}10 \not> {}^{-}2$ and $^{-}10 = {}^{-}2 + \underline{^{-}8}$

(6) $^{+}4 \not> {}^{+}10$ and $^{+}4 = {}^{+}10 + \underline{^{-}6}$

(7) $^{+}6 \not> {}^{+}6$ and $^{+}6 = {}^{+}6 + \underline{0}$

Record your results.

* * *

For each of the following exercises, circle [on your worksheet] either the '>' or the '$\not>$', and either the word 'positive' or the word 'nonpositive'.

(1) $^{+}9 \left\{ \begin{matrix} > \\ \not> \end{matrix} \right\} {}^{+}7$ and $^{+}9 = {}^{+}7 + \text{a} \left\{ \begin{matrix} \text{positive} \\ \text{nonpositive} \end{matrix} \right\}$ number

(2) $^{-}6 \left\{ \begin{matrix} > \\ \not> \end{matrix} \right\} {}^{-}11$ and $^{-}6 = {}^{-}11 + \text{a} \left\{ \begin{matrix} \text{positive} \\ \text{nonpositive} \end{matrix} \right\}$ number

(3) $^{+}10 \left\{ \begin{matrix} > \\ \not> \end{matrix} \right\} {}^{+}15$ and $^{+}10 = {}^{+}15 + \text{a} \left\{ \begin{matrix} \text{positive} \\ \text{nonpositive} \end{matrix} \right\}$ number

(4) $^{+}2 \left\{ \begin{matrix} > \\ \not> \end{matrix} \right\} {}^{+}2$ and $^{+}2 = {}^{+}2 + \text{a} \left\{ \begin{matrix} \text{positive} \\ \text{nonpositive} \end{matrix} \right\}$ number

Check your answers.

(1) $^{+}9 > {}^{+}7$ and $^{+}9 = {}^{+}7 +$ a *positive* number

(2) $^{-}6 > {}^{-}11$ and $^{-}6 = {}^{-}11 +$ a *positive* number

(3) $^{+}10 \not> {}^{+}15$ and $^{+}10 = {}^{+}15 +$ a *nonpositive* number

(4) $^{+}2 \not> {}^{+}2$ and $^{+}2 = {}^{+}2 +$ a *nonpositive* number

Record your results.

* * *

Work the following exercises as you did the preceding one.

(1) $^{+}1273 \left\{ \begin{matrix} > \\ \not> \end{matrix} \right\} {}^{-}84$ and $^{+}1273 = {}^{-}84 + \text{a} \left\{ \begin{matrix} \text{positive} \\ \text{nonpositive} \end{matrix} \right\}$ number

(2) $^{-}85 \left\{ \begin{matrix} > \\ \not> \end{matrix} \right\} {}^{-}84$ and $^{-}85 = {}^{-}84 + \text{a} \left\{ \begin{matrix} \text{positive} \\ \text{nonpositive} \end{matrix} \right\}$ number

(3) $^{-}58 \left\{ \begin{matrix} > \\ \not> \end{matrix} \right\} {}^{-}58$ and $^{-}58 = {}^{-}58 + \text{a} \left\{ \begin{matrix} \text{positive} \\ \text{nonpositive} \end{matrix} \right\}$ number

(4) $0 \left\{ \begin{matrix} > \\ \not> \end{matrix} \right\} {}^{+}4$ and $0 = {}^{+}4 + \text{a} \left\{ \begin{matrix} \text{positive} \\ \text{nonpositive} \end{matrix} \right\}$ number

Check your answers.

(1) $^{+}1273 > {}^{-}84$ and $^{+}1273 = {}^{-}84 +$ a *positive* number

(2) $^{-}85 \not> {}^{-}84$ and $^{-}85 = {}^{-}84 +$ a *nonpositive* number

(3) $^{-}58 \not> {}^{-}58$ and $^{-}58 = {}^{-}58 +$ a *nonpositive* number

(4) $0 \not> {}^{+}4$ and $0 = {}^{+}4 +$ a *nonpositive* number

Record your results.

* * *

The exercises on the last few pages suggest the following <u>definition</u> of the greater-than relation.

> A first real number is said to be greater than a second real number if and only if the first number is equal to the second number plus some positive number.

If someone didn't know anything about the greater-than relation for real numbers, this definition would tell him what it was <u>provided that</u> he already knew something about adding a positive number. For example, if he knew that

$$^{-}8 = {}^{-}10 + \text{positive } 2,$$

he could use the definition to learn that

$$^{-}8 \text{ is greater than } {}^{-}10.$$

On your work sheet write the greater-than statement which follows from the definition and the fact that

$$^{+}2 = {}^{-}100 + \text{positive } 102.$$

We give a definition of the

LESS - THAN RELATION

on page 42, and provide the students with considerable practice in using that definition.

[Part 115] [Page 42]

Check your answers.

The numeral for the larger number of each pair is circled.

(a) (+1), −1 (b) (+973.1), +973.01 (c) (−1), −7
(d) −16, (−15) (e) (+43), −93 (f) (0.0), −42
(g) 0, (+1) (h) −3, (0) (i) (+14), 0
(j) +1, +1 (k) −15, −15

In exercise (j), neither numeral names a larger number than the other. This is also the case in exercise (k). Consider yourself correct only if you didn't circle any of the numerals in (j) and (k).

Record your results.

* * *

Another relation among real numbers is the

LESS-THAN RELATION.

The less-than relation is usually defined in terms of the greater-than relation.

> A first number is less than a second number
> if and only if
> the second number is greater than the first number.

Fill the blanks on your work sheet.

(1) We know that +7 > +4. So, by the definition of the less-than relation, +4 is less than ___?___.

(2) −5 is less than −1 because ___?___ > ___?___.

Exercises like those on page 47 are designed to make the student aware of the difference between

THE GREATER - THAN RELATION
and
THE IS - NOT - LESS - THAN RELATION

and the difference between

THE LESS - THAN RELATION
and
THE IS - NOT - GREATER - THAN RELATION.

[Part 115] [Page 47]

Check your answers.

Numerals which would make true sentences are circled.

(1) +9 > ?	(−16)	(−14)	(−7)	(−1)	(0)	(+3)	(+4)	+9
(2) +3 > ?	(−16)	(−14)	(−7)	(−1)	(0)	+3	+4	+9
(3) 0 > ?	(−16)	(−14)	(−7)	(−1)	0	+3	+4	+9
(4) −1 > ?	(−16)	(−14)	(−7)	−1	0	+3	+4	+9
(5) −7 > ?	(−16)	(−14)	−7	−1	0	+3	+4	+9
(6) −16 > ?	−16	−14	−7	−1	0	+3	+4	+9

Record your results.

* * *

Just as we use '$\ngtr$' as an abbreviation for 'is not greater than', we will use '$\nless$' as an abbreviation for 'is not less than'. Thus, $+4 \nless +1$. [Read this as 'positive four is not less than positive one'.]

For each question, tell which of the real numbers listed at the right would make the incomplete sentence true. Circle the right answers on your work sheet. [There may be more than one correct choice.]

(1) +9 < ?	+11	+10	+9
(2) +9 $\ngtr$?	+11	+10	+9
(3) −4 < ?	+2	−3	−4
(4) −4 $\nless$?	+2	−3	−4
(5) +7 > ?	+5	+6	+7
(6) +7 $\ngtr$?	+5	+6	+7
(7) −8 > ?	−10	−9	−8
(8) −8 $\nless$?	−10	−9	−8

The awareness developed by exercises like those on page 47 prepares the way for the introduction of:

$\leq$

and:

$\geq$

on pages 51-53.

[Part 115] [Page 51]

Check your answers.

For each first real number and for each second real number,

either the first number $=$ the second number,

or the first number $>$ the second number,

or the first number $<$ the second number.

[The order of the 3 signs doesn't matter.]

Record your results.

* * *

Fill the blanks in these sentences.

(1) To say that

a first real number $\not>$ a second real number

amounts to saying that

the first number _?_ the second number

or the first number _?_ the second number.

(2) To say that

a first real number $/$ a second real number

amounts to saying that

the first number _?_ the second number

or the first number _?_ the second number.

[Part 115] [Page 52]

Check your answers.

(1) To say that

a first real number $\not>$ a second real number

amounts to saying that

the first number $<$ the second number or the first number $=$ the second number.

(2) To say that

a first real number $/$ a second real number

amounts to saying that

the first number $>$ the second number or the first number $=$ the second number.

Record your results.

* * *

The sentence:

the first number $<$ the second number or the first number $=$ the second number

can be abbreviated to:

the first number _ the second number

[Read the sign '_' as 'is less than or equal to'.]

In a similar manner, write on your work sheet an abbreviation for the sentence:

the first number > the second number or the first number = the second number

[Part 115] [Page 53]

Check your answer.

Three ways of saying the same thing:

the first number $\not<$ the second number

the first number $>$ the second number or the first number $=$ the second number

the first number _ the second number

ree ways of saying another thing:

the first number $/$ the second number

the first number $>$ the second number or the first number $=$ the second number

the first number $\geq$ the second number

Record your results.

* * *

On your work sheet, rewrite each of the following sentences in words. Then tell if the sentence is true or false by circling the 'T' or the 'F'.

Sample	Answers
(a) $\cdot 9 \not> \cdot 10$	·9 is not greater than ·10 (T) F
(b) $\cdot 9 \leq \cdot 10$	·9 is less than or equal to ·10 (T) F

(1) (a) $\cdot 8 \not< \cdot 2$	(2) (a) $^{-}6 \not< {}^{-}6$	(3) (a) $^{-}4 \not> {}^{-}4$
(b) $\cdot 8 \geq \cdot 2$	(b) $^{-}6 \leq {}^{-}6$	(b) $^{-}4 \leq {}^{-}4$

PART 116

Our development leading to the definition of

THE GREATER - THAN RELATION

for real numbers pay off in Part 116, where we quickly arrive at a definition of

THE GREATER - THAN RELATION

for the <u>numbers of arithmetic</u>.

[Part 116] [Page 3]

Check your answers.

(1) Here is our definition of the greater-than relation for real numbers:

> A first real number is greater than a second real number
> if and only if
> the first number is equal to the *second* number plus some *positive* number.

(2) $^{+}2 > {}^{-}4$ because $\left\{ \begin{array}{l} {}^{+}2 = 4 + \underline{{}^{+}6} \\ \text{and} \\ \underline{{}^{+}6} \text{ is a positive number.} \end{array} \right.$

(3) ${}^{+}6 \not> {}^{+}6$ because $\left\{ \begin{array}{l} {}^{+}6 = {}^{+}6 + \underline{0} \\ \text{BUT} \\ \underline{0} \text{ is NOT a positive number.} \end{array} \right.$

Record your results.

* * *

There is also a greater-than relation for the NUMBERS OF ARITHMETIC.

We say that

the NUMBER OF ARITHMETIC 9

is greater than

the NUMBER OF ARITHMETIC 3

because 9 = 3 + 6 and 6 is a NONZERO NUMBER OF ARITHMETIC.

Also, we say that 11 > 6 just because 11 = 6 + 5 and 5 ≠ 0.

Answer this question on your work sheet.

Definition of the greater-than relation for numbers of arithmetic:

> A first number of arithmetic is greater than a second number of arithmetic
> if and only if
> the first number is equal to the second number plus some __?__ number of arithmetic.

[Part 116] [Page 4]

Check your answer.

> A first number of arithmetic is greater than a second number of arithmetic
> if and only if
> the first number is equal to the second number plus some *nonzero* number of arithmetic.

Record your result.

* * *

This definition tells us that

12 > 8 because $\left\{ \begin{array}{l} 12 = 8 + 4 \\ \text{and} \\ 4 \neq 0. \end{array} \right.$

Fill in the blanks on your work sheet.

(1) 18 > 2 because $\left\{ \begin{array}{l} 18 = 2 + \underset{\cdot\cdot}{\overset{?}{\cdot}} \\ \text{and} \\ \underset{\cdot\cdot}{\overset{?}{\cdot}} \neq 0 \end{array} \right.$

(2) 23 > $\underset{\cdot\cdot}{\overset{?}{\cdot}}$ because $\left\{ \begin{array}{l} 23 = \underset{\cdot\cdot}{\overset{?}{\cdot}} + 10 \\ \text{and} \\ 10 \neq \underline{?} \end{array} \right.$

(3) $\underset{\sim}{7}$ > $\underset{\cdot\cdot}{5}$ because $\left\{ \begin{array}{l} \overset{?}{\sim} = \underset{\cdot\cdot}{\overset{?}{\cdot}} + \underline{\ ?\ } \\ \text{and} \\ \underline{\ ?\ } \neq \underline{\ ?\ } \end{array} \right.$

[Part 116] [Page 5]

Check your answers.

(1) 18 > 2 because { 18 = 2 + 16 and 16 ≠ 0

(2) 23 > 13 because { 23 = 13 + 10 and 10 ≠ 0

(3) 7 > 5 because { 7 = 5 + 2 and 2 ≠ 0

Record your results.

* * *

There is also a less-than relation for the numbers of arithmetic. Recall that for the real numbers we defined:

> A first real number is less than a second real number if and only if the second number is greater than the first.

Similarly we define:

> A first number of arithmetic is less than a second number of arithmetic if and only if the second number is ___?___ ___?___ the first.

Fill the blanks on your work sheet.

[Part 116] [Page 6]

Check your answers.

> A first real number is less than a second real number if and only if the second number is greater than the first.

Similarly:

> A first number of arithmetic is less than a second number of arithmetic if and only if the second number is greater than the first.

Record your results.

* * *

Answer the following questions.

(1) 2 < 18 because 18 ? 2

(2) 13 < 23 because ? > ?

THE GREATER - THAN RELATION for numbers of arithmetic is necessary to the introduction of the idea comparing numbers of arithmetic by means of a picture of a ray. This idea will be useful later when we wish to suggest the use of a picture of a line in helping to compare real numbers.

[Part 116] [Page 8]

Check your answers.

(1) 0, ⅓, 2, 6.5, 11, 17

(2) 0, 1, 2, 3, 4, 5

Record your results.

* * *

Many people think of the numbers of arithmetic as being "lined up in order", starting with the smallest number of arithmetic, and with greater numbers to the "right" of smaller numbers.

What is the smallest number of arithmetic?

Write the answer on your work sheet.

[Part 11a] [Page 9]

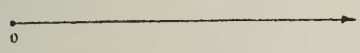
Check your answer.

The smallest number of arithmetic is 0.

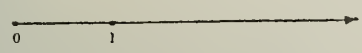
Record your result.

* * *

We can show the "line-up" of the numbers of arithmetic by drawing a ray and marking the end point with a '0'.



Then, since 1 is greater than 0 we label a point on the ray to the right of '0' with a '1'.

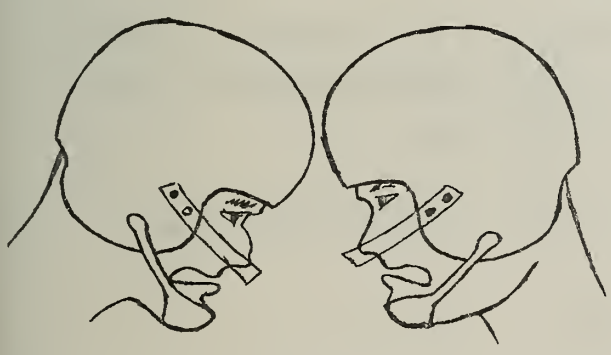


Numbers larger than 1 will correspond to points on the ray to the right of the point for 1.

On your work sheet mark dots for 2, $1\frac{1}{2}$, $2\frac{1}{2}$, 3, and $3\frac{1}{2}$. Label the four dots by writing a numeral under each dot.

The idea of using a picture of a line to help in comparing real numbers follows from the problem which begins on page 15.

[Part 11b] [Page 15]



The Zabranchburg Zebras and the Anabru Bruins were tied for first place in the county football conferences. Two days before the Zebras were to meet the Bruins, for the championship game, Z. H. S. Coach F. W. Neal held a special practice session in order to choose someone as starting fullback against the Bruins. There were seven candidates for the position. Coach Neal decided to have each candidate run four plays in scrimmage and record each player's gains or losses in yardage with real numbers. Thus he would write:

+5

when he meant:

a gain of 5 yards.

Similarly, he would write:

8

when he meant:

a ___?___ of 8 yards

and:

___?___

when the play resulted in neither a gain nor a loss.

Fill in the blanks on your work sheet.

[Part 11b] [Page 16]

Check your answers.

He would write ' 8' when he meant 'a loss of 8 yards' and ' 0 ' when the play resulted in neither a gain nor a loss.

Record your results.

* * *

Coach Neal asked the Z. H. S. football manager Wilbur Williams to keep a record of each player's outcomes on each of his four plays. Wilbur's record looked like this.

	1st Play	2nd Play	3rd Play	4th Play	Total yardage gained or lost
Brown	+5.5	1.5	+3	+2	+9
Hale	+7.5	+4	7	4.5	?
Harris	+2	3	+13	−1	?
Hart	9	0	+11	+1	?
Hoffman	7	3	0	+9	?
Kanski	0	0	−4	0	?
Zwoyer	+5	6	+3	+3	?

Complete the chart on your work sheet.

[Part 116] [Page 17]

Check your answers.

	1st Play	2nd Play	3rd Play	4th Play	Total yardage gained or lost
Brown	+5.5	1.5	+3	+2	+9
Hale	+7.5	+4	7	4.5	0
Harris	+2	3	+13	1	+11
Hart	9	0	+11	+1	+3
Hoffman	-7	-3	0	+9	-1
Kanski	0	0	-4	0	-4
Zwoyer	+5	-6	+3	+3	+5

Record your results.

* * *

Wilbur now decided to compare the playing ability of the candidates by "lining up" their real number outcomes like this:

-4 , -1 , ? , ? , +5 , ? , ?

worst outcome — best outcome

poorest player — best player

Kanski , Hoffman , ? , ? , Zwoyer , ? , ?

Fill in the blanks on your work sheet.

[Part 116] [Page 18]

Check your answers.

-4 , -1 , 0 , +3 , +5 , +9 , +11

worst outcome — best outcome

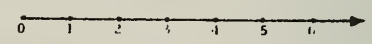
poorest player — best player

Kanski , Hoffman , Hale , Hart , Zwoyer , Brown , Harris

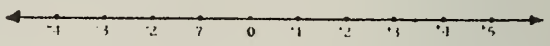
Record your results.

* * *

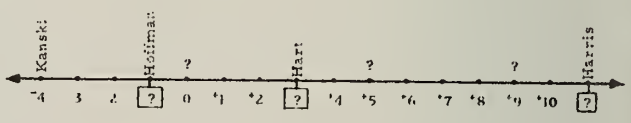
After completing this "lining up" process it occured to Wilbur that this process was similar to a process of "lining up" numbers of arithmetic that he had learned in his math class. Just as the greater-than relation for NUMBERS OF ARITHMETIC made it easy to think the numbers of arithmetic by drawing a picture of a RAY:



so, the greater-than relation for the REAL NUMBERS makes it easy to think of the real numbers by drawing a picture of a line:



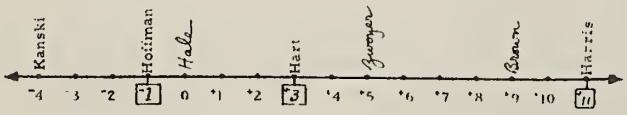
Here is the picture that Wilbur used to compare the performances of the seven candidates



Complete the drawing on your work sheet.

[Part 116] [Page 19]

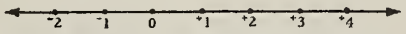
Check your answers.



Record your results.

* * *

When you draw a picutre like:



you have a picture of (part of) the real number line.

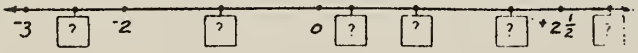
[Since it is impossible to draw a picture of all of the real number line, let's agree to say:

a picture of the real number line

when we mean to say:

a picture of part of the real number line]

On the picture of the real number line below, estimate the numbers that are represented by the dots and fill the blanks on your work sheet.



The use of pictures of rays and lines to aid in comparing numbers leads to the defining of:

THE NUMBER RAY

and:

THE NUMBER LINE

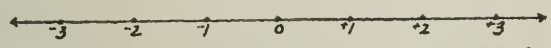
on page 25.

Pictures of THE NUMBER LINE are helpful in thinking about many relationships between real numbers. Some of these are suggested on pages 32-34.

[Part 116] [Page 25]

Check your answers.

Similarly, because the set of real numbers can be visualized by thinking of a picture of a line:



we sometimes call the set of real numbers THE REAL NUMBER LINE or, simply, THE NUMBER LINE.

Record your results.

* * *

A mental picture of the number line makes it easy to compare real numbers.

For each pair of real numbers listed below, tell whether the first is less than the second or the first is greater than the second.

(1) (⁺3, ⁺17) (2) (⁻6, ⁻5)

(3) (⁺0.0012, ⁻0.0138) (4) ($^{+}\frac{11}{3}$, $^{+}\frac{21}{6}$)

(5) (⁺152, ⁻2176) (6) (⁻0.00016, ⁻43213)

[Part 116] [Page 32]

Check your answers.

(1) $^{+}6 \neq {}^{-}3$, $^{+}6 > {}^{-}3$, $^{+}6 \not< {}^{-}3$, $^{+}6 \geq {}^{-}3$

(2) $^{-}10 \neq {}^{-}9$, $^{-}10 < {}^{-}9$, $^{-}10 \not> {}^{-}9$, $^{-}10 \leq {}^{-}9$

(3) $^{-}\frac{10}{3} \neq {}^{+}2$, $^{-}\frac{10}{3} < {}^{+}2$, $^{-}\frac{10}{3} \not> {}^{+}2$, $^{-}\frac{10}{3} \leq {}^{+}2$

(4) $0 \neq {}^{-}\frac{1}{782}$, $0 > {}^{-}\frac{1}{782}$, $0 \not< {}^{-}\frac{1}{782}$, $0 \geq {}^{-}\frac{1}{782}$

Record your results.

* * *

You have seen that the relation greater-than makes it possible for us to think of the set of real numbers as being lined-up. So, we talk about the set of real numbers as the number line, and talk about real numbers as points of the number line.

This geometric way of talking about real numbers is very useful because it helps us visualize facts about the real numbers. For example, we can talk about a real number being between two other real numbers just as if the real numbers were dots on a picture of a line.

Circle the correct answers on your work sheet. [There may be more than one right answer for a given question.]

(1) ⁺3 is between ___?___ (A) ⁺1 and ⁺7 (B) ⁻1 and ⁻4 (C) ⁻2 and ⁺5

(2) ⁻7 is between ___?___ (A) ⁻8 and ⁺9 (B) ⁻6 and ⁺4 (C) ⁺3 and ⁻9

(3) ⁻2 is between ___?___ (A) 0 and ⁺5 (B) 0 and ⁻5 (C) ⁻1 and 3

(4) ⁺6 is between ___?___ (A) ⁺4 and ⁺5 (B) ⁺5 and ⁺6 (C) ⁺4 and ⁺7

(5) ⁻3.01 is between ___?___ (A) ⁻2 and ⁻4 (B) ⁻3 and ⁻2 (C) ⁻3 and ⁻4

[Part 116] [Page 33]

Check your answers.

(1) ⁺3 is between ___?___ (A) ⁺1 and ⁺7 (B) ⁻1 and ⁻4 (C) ⁻2 and ⁺5

(2) ⁻7 is between ___?___ (A) ⁻8 and ⁺9 (B) ⁻6 and ⁺4 (C) ⁺3 and ⁻9

(3) ⁻2 is between ___?___ (A) 0 and ⁺5 (B) 0 and ⁻5 (C) ⁻1 and ⁻3

(4) ⁺6 is between ___?___ (A) ⁺4 and ⁺5 (B) ⁺5 and ⁺6 (C) ⁺4 and ⁺7

(5) ⁻3.01 is between ___?___ (A) ⁻2 and ⁻4 (B) ⁻3 and ⁻2 (C) ⁻3 and ⁻4

Record your results.

* * *

Another example of a geometric way of thinking of real numbers is to talk about a real number being closer to one real number than to another.

Circle the correct answers on your work sheet. [There may be more than one right answer for a given question.]

(1) ⁺7 is closer to ⁺9 than it is to ___?___. (A) ⁺11 (B) ⁺8 (C) ⁺4

(2) ⁺3 is closer to 0 than it is to ___?___. (A) ⁺8 (B) 3 (C) ⁺5

(3) 5 is closer to ⁺1 than it is to ___?___. (A) ⁻12 (B) ⁻11 (C) ⁻10

(4) 2 is as close to ⁻8 as it is to ___?___. (A) ⁺3 (B) ⁺8 (C) ⁺4

[When we say 'as close to' we mean 'exactly as close to'.]

[Part 116] [Page 34]

Check your answers.

(1) ⁺7 is closer to ⁺9 than it is to ___?___. (A) ⁺11 (B) ⁺8 (C) ⁺4

(2) ⁺3 is closer to 0 than it is to ___?___. (A) ⁺8 (B) 3 (C) ⁺5

(3) 5 is closer to ⁺1 than it is to ___?___. (A) ⁻12 (B) ⁻11 (C) ⁻10

(4) ⁻2 is as close to 8 as it is to ___?___. (A) ⁺3 (B) ⁺8 (C) ⁺4

Record your results.

* * *

Write the answers to these questions on your work sheet. [Remember that 'as close to' means 'exactly as close to'.]

(1) ⁺6 is as close to ⁺4 as it is to ___?___.

(2) ⁻5 is as close to ⁻3 as it is to ___?___.

(3) ⁻4 is as close to ⁺5 as it is to ___?___.

(4) ⁺3 is as close to ⁻2 as it is to ___?___.

(5) 0 is as close to ⁺5 as it is to ___?___.

(6) ⁺6 is as close to ⁻1 as it is to ___?___.

(7) 1 is as close to ⁺6 as it is to ___?___.

In particular, we use our pictures of THE NUMBER LINE to give meaning to:

the distance between real numbers

[Part 116] [Page 35]

Check your answers.

(1) $^+6$ is as close to $^+4$ as it is to $^+8$

(2) $^-5$ is as close to $^-3$ as it is to $^-7$

(3) $^-4$ is as close to $^+5$ as it is to $^-13$

(4) $^+3$ is as close to $^-2$ as it is to $^+8$

(5) 0 is as close to $^+5$ as it is to $^-5$

(6) $^+6$ is as close to $^-1$ as it is to $^+13$

(7) 1 is as close to $^+6$ as it is to $^-8$

Record your results.

* * *

Another geometric idea which is related to the idea of a real number being as close to one real number as it is to another is the idea of the distance between real numbers. We can say, for example, that the distance between the real numbers $^+2$ and $^+5$ is the same as the distance between the real numbers $^+23$ and $^+20$ because $^+2$ is as close to $^+5$ as $^+23$ is to $^+20$.

Circle the correct answers on your work sheet. [There may be more than one right answer for a given question.]

(1) The distance between $^+5$ and $^+10$ is the same as the distance between ___?___.
(A) $^+9$ and $^+12$ (B) $^-3$ and $^-8$ (C) $^-10$ and $^-5$ (D) $^+7$ and $^+1$

(2) The distance between $^-7$ and $^-3$ is the same as the distance between ___?___.
(A) $^+8$ and $^+12$ (B) $^+12$ and $^+8$ (C) $^-3$ and $^-7$ (D) $^-7$ and $^-10$

[Part 116] [Page 36]

Check your answers.

(1) The distance between $^-5$ and $^+10$ is the same as the distance between ____.
(A) $^+9$ and $^+12$ (B) $^-3$ and $^-8$ (C) $^-10$ and $^-5$ (D) $^+7$ and $^+1$

(2) The distance between $^-7$ and $^-3$ is the same as the distance between ____.
(A) $^+8$ and $^+12$ (B) $^+12$ and $^+8$ (C) $^-3$ and $^-7$ (D) $^-7$ and $^-10$

Record your results.

* * *

For convenience, let's agree to abbreviate the phrase:

the distance between $^-3$ and $^-10$

by writing:

$\mathcal{D}(^-3, {}^-10)$

So, when you see, for example: $\mathcal{D}(^-5, {}^+6)$

you should think: the distance between $^-5$ and $^+6$

Circle the correct answers on your work sheet. [There may be more than one right answer for a given question.]

(1) $\mathcal{D}(^+2, {}^+8) =$ ___?___ (A) $\mathcal{D}(^+7, {}^+13)$ (B) $\mathcal{D}(^+8, {}^+2)$ (C) $\mathcal{D}(0, {}^+6)$

(2) $\mathcal{D}(^-3, 0) =$ ___?___ (A) $\mathcal{D}(0, {}^-3)$ (B) $\mathcal{D}(^-4, {}^+7)$ (C) $\mathcal{D}(0, {}^+3)$

(3) $\mathcal{D}(^-2, {}^+5) =$ ___?___ (A) $\mathcal{D}(^+8, {}^-1)$ (B) $\mathcal{D}(^+8, {}^+1)$ (C) $\mathcal{D}(^+5, {}^-2)$

(4) $\mathcal{D}(^+8, 3) =$ ___?___ (A) $\mathcal{D}(0, {}^-11)$ (B) $\mathcal{D}(^+2, {}^+9)$ (C) $\mathcal{D}(0, {}^+11)$

(5) $\mathcal{D}(^+39, {}^-58) =$ ___?___ (A) $\mathcal{D}(0, {}^+21)$ (B) $\mathcal{D}(^-17, 0)$ (C) $\mathcal{D}(^-58, {}^+3$

(6) $\mathcal{D}(^+9, {}^-7) =$ ___?___ (A) $\mathcal{D}(^+9, {}^+7)$ (B) $\mathcal{D}(^+7, {}^+9)$ (C) $\mathcal{D}(^-7, {}^+9)$

The mapping of the real numbers onto the numbers of arithmetic, displayed on page 43, is the end product of our consideration of

the distance between real numbers.

This type of display is new to the student, but its meaning should be clear since it is merely a pictorial summary of a relationship between two sets of which the student is already aware.

Elements in the range of the correspondence suggested on page 43 are each given a name on page 47, and function notation is introduced on page 50.

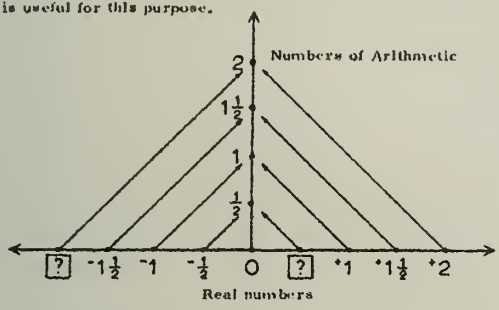
[Part 116] [Page 43]

Check your answer.

No matter what first and second real numbers you pick, if the first plus the second is 0 then the second is the *opposite* of the *first*. Record your result.

* * *

So, to find the distance between a first real number and a second real number, just subtract either one from the other, and then find the number of arithmetic which corresponds to the difference. A picture like this one is useful for this purpose.

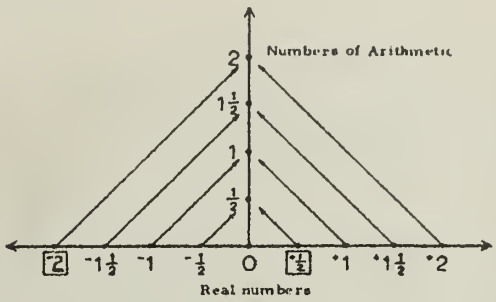


The picture shows us that the real number ⁺1 corresponds to the number of arithmetic 1, the real number ⁻1 corresponds to the number of arithmetic 1 and the real number 0 corresponds to the number of arithmetic 0.

Complete the picture above by filling in the boxes on your work sheet.

[Part 116] [Page 44]

Check your answers.



Record your results.

* * *

Sample. Find the distance between ⁺2 and 1.

Solution. ⁺2 - ⁻1 = ⁺3 [or ⁻1 - ⁺2 = 3]

Since ⁺3 [or ⁻3] corresponds to 3 in the picture, it follows that $\mathcal{D}$ (⁺2, ⁻1) = 3.

The diagram above will help you complete each of these statements on your work sheet..

(1) The distance between ⁺3 and ⁺8 is the number of arithmetic which corresponds to ⁺3 - _?_ and to ⁺8 - ?. This number is _?_.

(2) The distance between 7 and ⁺9 is the _?_ _?_ _?_ which corresponds to ⁻7 - _?_. This number is ?.

(3) The distance between ⁻7 and ⁺9 is the _?_ _?_ _?_ which corresponds to ⁺9 - _?_. This number is ?.

[Part 116] [Page 47]

Check your answer.

Since we have agreed that distance is measured by a NUMBER OF ARITHMETIC, Arthur realized that both boys are correct as far as they go, but that after they find the differences between the two chosen real numbers they should then find the number of arithmetic to which these differences correspond. this number of arithmetic is the distance between Ruth's number and Rachel's number.
[You are correct if you wrote something like this.]

Record your result.

* * *

Instead of saying a long phrase like:

the number of arithmetic which corresponds with ⁻5 let us say:

the arithmetic value of ⁻5

[pronounce 'arithmetic value' as 'arithMETic value']

For example if you see the phrase:

the arithmetic value of ⁻8

just think: The number of arithmetic which corresponds with 8. And, since this number of arithmetic is 8, it follows that the arithmetic value of ⁻8 is 8.

Write the answers to these questions on your work sheet.

(1) What number of arithmetic is the arithmetic value of ⁺6?

(2) What number of arithmetic is the arithmetic value of ⁻2?

[Part 116] [Page 50]

Check your answers.

(1) The arithmetic value of ⁺3 = 3.

(2) The arithmetic value of (⁺9 - ⁺6) = 3.

(3) The arithmetic value of (⁺6 - ⁺9) = 3.

(4) The distance between ⁺9 and ⁺6 is the *arithmetic value* of (⁺6 - ⁺9).

(5) The arithmetic value of ⁺817 = 817.

(6) The distance between 0 and ⁺817 = 817.

(7) The arithmetic value of ⁻401 = 401.

(8) The distance between 0 and ⁻401 = 401.

Record your results.

* * *

Just as we agreed to abbreviate the phrase:

the distance between ⁻3 and ⁻10

by writing:

$\mathcal{D}$ (⁻3, ⁻10)

Let's agree to abbreviate the phrase:

the arithmetic value of ⁺3

by writing:

$\mathcal{A}$ (⁺3)

Fill in the blanks on your work sheet.

(1) $\mathcal{A}$ (⁺6) = ? (2) $\mathcal{A}$ (6) = ? (3) $\mathcal{A}$ (0) = ?

(4) $\mathcal{A}$ (⁺5) = ? (5) $\mathcal{A}$ (⁺5 - ⁺9) = ?

The set of ordered pairs suggested by the correspondence (page 52) is given a name on page 53.

Our treatment of ARITHMETIC VALUING allows us to use a diagram to introduce a new correspondence on page 58. We immediately extract a list of ordered pairs and give the new operation a name. Function notation for the operations appears on page 63.

[Part 116] [Page 52]

Check your answers.

(1) $A(^{+}3 - {}^{+}4) = A(^{-}1) =$ 1

(2) $A(^{+}4 - {}^{+}3) =$ 1

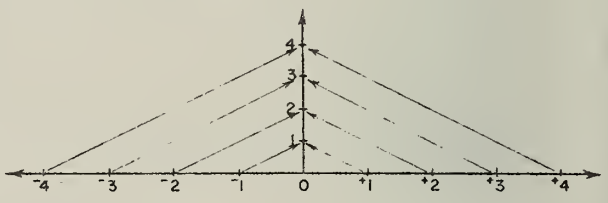
(3) $A(^{-}7 - {}^{-}9) =$ 2

(4) $A(^{-}9 - {}^{-}7) =$ 2

Record your results.

* * *

From the diagram:



we get the list of pairs:

$(^{-}4, 4)$ $(^{-}3, 3)$ $(^{-}2, 2)$ $(^{-}1, 1)$ $(0, 0)$
$(^{+}4, 4)$ $(^{+}3, 3)$ $(^{+}2, 2)$ $(^{+}1, 1)$
. . .

What is a good name for this list?

Write your answer on your work sheet.

[Part 116] [Page 53]

A good name for this list is

arithmetic Valuing

Check your answers.

$(^{-}4, 4)$ $(^{-}3, 3)$ $(^{-}2, 2)$ $(^{-}1, 1)$ $(0, 0)$
$(^{+}4, 4)$ $(^{+}3, 3)$ $(^{+}2, 2)$ $(^{+}1, 1)$
. . .

Record your results.

* * *

Answer the following questions on your work sheet.

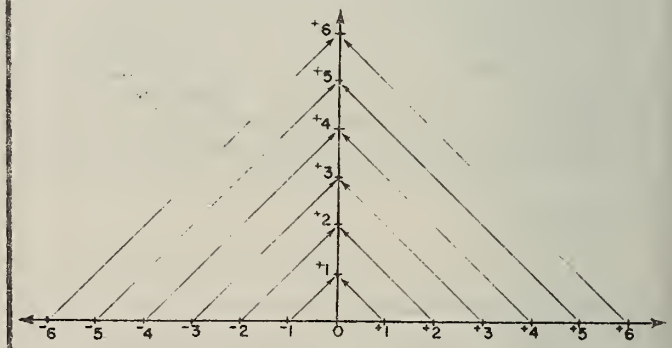
Here is another list of pairs which belong to arithmetic valuing. Fill in the blanks.

arithmetic valuing

$(^{+}17,$? $)$ $(^{+}\frac{1}{2},$? $)$
$(^{+}3, 3)$ $(^{-}3,$? $)$ $(^{-}6.28,$? $)$
$(^{+}14.03,$? $)$ $($? $, 0)$

[Part 116] [Page 58]

The diagram below shows how we might think about a correspondence between all of the reals and the nonnegative reals.



From this diagram, we can see that

3 and $^{+}3$ both correspond to $^{+}3$.

Using the diagram, complete the following statements on your work sheet.

(1) ? and $^{+}2$ both correspond to $^{+}2$.

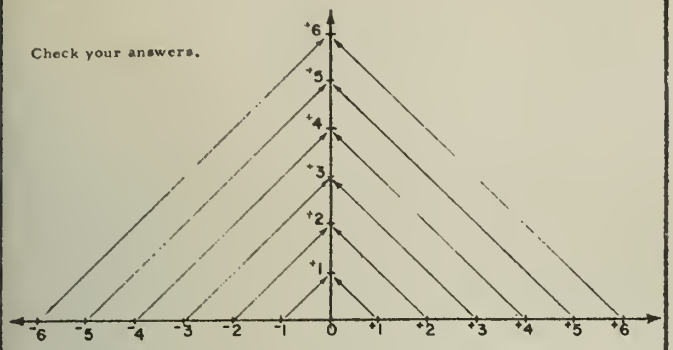
(2) $^{-}7$ and ? both correspond to ? .

(3) $^{-}3\frac{1}{2}$ corresponds to ? .

(4) Both $\frac{^{+}3}{4}$ and $\frac{^{-}3}{4}$ correspond to ? .

[Part 116] [Page 59]

Check your answers.



(1) 2 and +2 both correspond to +2.

(2) -7 and +7 both correspond to +7.

(3) $-3\frac{1}{2}$ corresponds to $+3\frac{1}{2}$.

(4) Both $\frac{+3}{4}$ and $\frac{-3}{4}$ correspond to $\frac{+3}{4}$.

Record your results.

* * *

The picture above suggests a list of pairs of real numbers like the one below:

(+3, +3) (-3, +3)
(0, 0) (+1, +1) (-1, +1)
(-3.5, +3.5) (+2.2, +2.2)
...

Do you think that this list belongs to an operation? Circle the 'YES' or the 'NO' on your work sheet.

[Part 116] [Page 60]

Check your answers.

(+3, +3) (-3, +3)
(0, 0) (+1, +1) (-1, +1)
(-3.5, +3.5) (+2.2, +2.2)
...

yes, the list above belongs to an operation because there are no two pairs with the same first component.

Record your results.

* * *

Since the list above forms an operation it would be useful to have a name for the operation. We call this operation:

ABSOLUTE VALUING

(1) Here is another list which belongs to the operation ABSOLUTE VALUING.

(+1, +1) (-1, +1)
(+2.6, ?) (-2.6, ?)
(0, ?) (+192, ?)
(-192, ?)
...

Fill the blanks on your work sheet.

(2) Does the operation ABSOLUTE VALUING have an inverse? Circle the 'YES' or the 'NO'.

(3) (a) If your answer to question (2) was NO then give 2 pairs belonging to the list which show you are correct.
(b) If your answer to question (2) was YES, then give a good name for this operation other than 'the inverse of absolute valuing'.

[Answer question 3a or 3b, but not both.]

[Part 116] [Page 63]

Check your answers.

(1) The absolute value of (-5 + +7) is +2.

(2) The absolute value of (+6 + +1) is +7.

(3) The absolute value of (-7 + -2) is +9.

(4) The absolute value of (-2 + +7) is +5.

Record your results.

* * *

Instead of writing:

The absolute value of -20

let's write:

$$|-20|$$

[Read '|-20|' as you would read 'the absolute value of -20.']

Fill the blanks on your work sheet.

(1) The absolute value of (-5 + +12) is ? .

(2) The absolute value of (8 + -5) is ? .

(3) | 5 + +12 | = ?

(4) | 8 + 5 | = ?

Exercises like those on page 68 follow the introduction of the function notation and are designed to help the student discover helpful generalizations. These generalizations are soon put to work in solving the worded problems on pages 77 and 78.

[Part 116] [Page 68]

Check your answers.

(1) $|^{+}6| - |^{+}3| =$ $^{+}3$ (2) $|^{+}6 - {}^{+}3| =$ $^{+}3$

(3) $|^{-}6| - |^{+}3| =$ $^{+}3$ (4) $|^{-}6 - {}^{+}3| =$ $^{+}9$

(5) $|^{+}6| - |^{-}3| =$ $^{+}3$ (6) $|^{+}6 - {}^{-}3| =$ $^{+}9$

(7) $|^{-}6| - |^{-}3| =$ $^{+}3$ (8) $|^{-}6 - {}^{-}3| =$ $^{+}3$

Record your results.

* * *

Answer these questions.

(1) $|^{+}2| \times |^{+}9| =$? (2) $|^{+}2 \times {}^{+}9| =$?

(3) $|^{+}2| \times |^{-}9| =$? (4) $|^{+}2 \times {}^{-}9| =$?

(5) $|^{-}2| \times |^{+}9| =$? (6) $|^{-}2 \times {}^{+}9| =$?

(7) $|\ 2| \times |\ 9| =$? (8) $|^{-}2 \times {}^{-}9| =$?

[Part 116] [Page 77]

Check your answers.

(1) $|\ ?\ | + {}^{+}4 = {}^{+}9$; $^{+}5$ or $^{-}5$ (2) $^{+}9 \times |\ ?\ | = {}^{+}36$; $^{+}4$ or $^{-}4$

(3) $|\ ?\ + {}^{-}7| = {}^{+}5$; $^{+}2$ or $^{+}12$ (4) $|^{-}3 - \ ?\ | = {}^{+}2$; $^{-}1$ or $^{-}5$

(5) $^{+}7 + |\ ?\ | = {}^{+}6$; no such number [The absolute value of each real number is a nonnegative real number. There is no nonnegative real number which you can add to $^{+}7$ to get $^{+}6$.]

Record your results.

* * *

(1) Morton chooses a positive real number. He whispers his number to Fred. He whispers the ABSOLUTE VALUE of the number he chose to Allen. Compare the number Morton whispered to Fred with the number Morton whispered to Allen by using one of the symbols, '=', '<', or '>'.

(2) Frank chooses a negative real number. He whispers his number to Fred. He whispers the ABSOLUTE VALUE of the number he chose to Allen. Compare the number Frank whispered to Fred with number Frank whispered to Allen by using one of following symbols, '=', '<', or '>'.

[Part 116] [Page 78]

Check your answers.

(1) Fred's number = Allen's number; since no matter what positive real number you pick the ABSOLUTE VALUE of that positive number is that positive number.

(2) Fred's number < Allen's number; since no matter what negative real number you pick, the ABSOLUTE VALUE of that number is positive, and a negative number is always less than any positive number.

Record your results.

* * *

Ed picks a positive real number and Sam picks a negative real number. Each whispers his number to Karl. Karl finds the product of the two numbers and writes it down. Then Ed and Sam each whispers the absolute value of his real number to Lou. Lou finds the product of these two numbers and writes it down.

Complete the following to make true sentences by putting one of the signs '=', '>' or '=' in place of the question mark.

(1) Karl's number ? Lou's number

(2) |Karl's number| ? Lou's number

PARTS 41, 42, & 43

Parts 41, 42, and 43 were designed to present material from UICSM, Unit 4, as a special study on the discovery method. One group of students used Parts 41, 42, and 43 while another group used Parts 41, 42, and 43 [note the underlined numerals.] These latter named parts were designed to present the same material as that in Parts 41, 42, and 43, but on what one might call an expository model. However, no sample expository pages are exhibited here since the only essential difference between the two treatments is in the order of certain frames within each teaching sequence. In the discovery treatment, the frames in which verbalization of a concept or generalization occur are near the end of each teaching sequence, while in the expository treatment, these frames are at the beginning of the sequence.

Parts 41, 42, and 43, were written first. Each teaching sequence has three subsequences in this order:

1. a sequence of situations, examples, and exercises designed to develop a given concept or generalization;
2. a statement of the given concept or generalization;
3. two short sets of exercises involving the use of the given concept or generalization.

Parts 41, 42, and 43 were obtained from Parts 41, 42, and 43 by reordering the three subsequences of each teaching sequence listed above to obtain this ordering:

1. a statement of the given concept;
2. two short sets of exercises;
3. a series of situations, examples and exercises.

The first concept developed in Part 41 is that of union of sets. The program introduces a situation in which student committees serve as examples of sets. The union of two of these sets is the set which results when two committees hold a joint meeting. The programer had two main objectives in mind when inventing the committees and the membership of each: One of these was that committees should be so constituted that various cases could be presented such as; the union of disjointed sets, the union of sets with some but not all elements in common, the union of a set with one of its proper subsets, the union of a set with itself, and the union of a set with the empty set. The other objective was that no two students should have the same first letter in their first name. This was done to allow for the eventual listing of a given student by using a single symbol. Thus, while the Food Committee is originally listed like this:

Food Committee: Don, Hal, Laura, Nancy,

eventually it is listed like this:

{D, H, L N}.

An achievement test and a transfer test were constructed to cover Parts 41, 42, and 43. Items on the transfer test are probably of most interest for purposes of illustration. For this reason, only the first sample item below is taken from the achievement test. The remaining items are samples from the transfer test. [Correct choices are indicated with a small arrow.]

1. $\{2, 3, 4\} \cap R = \{2, 3\}$ if R = _____.

(A) $\{4\}$ (B) $\emptyset$

→ (C) $\{-2, -1\ 0, 1, 2, 3\}$ (D) $\{-4, 2, 3, 4\}$

2. Suppose that X, Y, and Z are sets and e belongs to X, e belongs to Y, but e does not belong to Z. It follows that _____.

(A) e belongs to $(X \cap Y) \cap Z$

(B) e belongs to $(X \cap Y) \cup Z$

(C) e does not belong to $X \cup (Y \cup Z)$

(D) e does not belong to $X \cap (Y \cup Z)$

3. Suppose that R, S, and T are sets and x belongs to $R \cap (S \cup T)$. It follows that _____.

(A) x belongs to $(R \cap S) \cap (R \cap T)$

(B) x belongs to $(R \cap S) \cap T$

(C) x belongs to $(R \cap S) \cup (R \cap T)$

(D) x belongs to $(R \cup S) \cap T$

4. Suppose that both (u, v) and (v, u) belong to one of the sets listed below. Which one is it?

(A) $\{(x, y),\ x \text{ and } y \text{ integers: } y = |x| + 3\}$

(B) $\{(x, y),\ x \text{ and } y \text{ integers: } x + 2y = 7$

(C) $\{(x, y),\ x \text{ and } y \text{ integers: } x^2 + y^2 = 25\}$

(D) $\{(x, y),\ x \text{ and } y \text{ integers: } 2 < x < 4 \text{ and } -4 < y < -2\}$

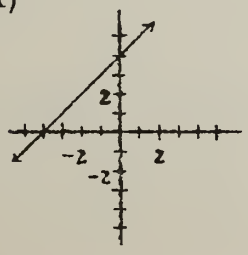
5. Suppose that X is the set of three integers 1, 2, and 3. This set has eight <u>subsets</u> A, B. C, D . . . , H. It follows that

$$(\ldots(((A \cup B) \cup C) \cup D)\ldots) \cup H) = _____.$$

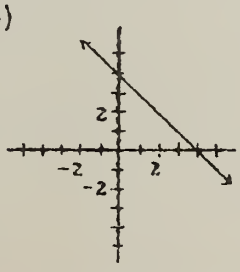
(A) $\emptyset$ (B) $\{1, 2, 3\}$ (C) $\{9\}$ (D) $\{1\}$

6. Consider sets of ordered pairs of <u>real numbers</u>, not just of integers. Such sets can be graphed on a picture of the <u>number plane</u>. Which of the following is a graph of $\{(x, y),\ x \text{ and } y \text{ real numbers: } y = x + 4\}$?

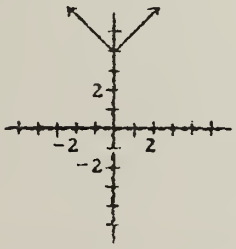
(A)



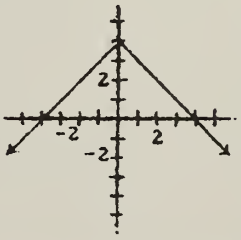
(B)



(C)



(D)



The concept to be developed is that of union. Note the following about the subcommitttees listed:

(1) there are pairs with no members in common (e.g., Food, Finance),
(2) there are pairs with some members in common (e.g., Food, Games),
(3) the Tickets Committee is a proper subset of the Finance Committee,
(4) the Prizes Committee and the Side Show Committee are the same set of students,
(5) the Rides Committee is the empty set.

* * *

The exercises on page 9 and those on page 10 are designed to do no more than focus attention on the various subcommittees and their members.

[Part 41] [Page 9]

The freshman class at Zabranchburg High School is planning a carnival. Each class member had a chance to volunteer for any of the subcommittees of the Carnival Committee. No two of the students who signed up for committee work have the same first name, so we can list the subcommittee members by using only first names. If the same name appears on more than one subcommittee list, that just means that the same student is on more than one subcommittee.

Here are lists of the carnival subcommittees.

Construction	Food	Games	Finance
Al	Don	Al	George
Bill	Hal	Hal	Julie
Don	Laura	Laura	Margie
Frank	Nancy	Pam	Sue

Tickets	Prizes	Side Show	Rides
George	Charles	Charles	[No one
Margie	Ed	Ed	volunteered.]
	Julie	Julie	
	Kathy	Kathy	

Answer these questions on your work sheet.

(1) How many subcommittees is Bill on?
(2) How many subcommittees is Laura on?
(3) One student is on three subcommittees. Which student is it?
(4) How many students are on the Finance Subcommittee?
(5) How many students are on the Rides Subcommittee?

[Part 41] [Page 10]

Check your answers.

Construction	Food	Games	Finance
Al	Don	Al	George
Bill /	Hal	Hal	Julie ///
Don	Laura //	~~Laura~~	Margie
Frank	Nancy	Pam	Sue

Tickets	Prizes	Side Show	Rides
George	Charles	Charles	[No one
Margie	Ed	Ed	volunteered.]
	~~Julie~~	~~Julie~~	
	Kathy	Kathy	

(1) Bill is on 1 subcommittee.
(2) Laura is on 2 subcommittees.
(3) Julie is on three subcommittees.
(4) 4 students are on the Finance Subcommittee.
(5) 0 students are on the Rides Subcommittee. [Your answer is also correct if you wrote a 'no' in the blank.]

Record your results in the result-check column on your work sheet.

* * *

Ten students are on more than one subcommittee. They are: Al, Don, Hal, Laura, George, Julie, Margie, Charles, Ed, and Kathy. Answer these questions on your work sheet.

(1) How many students are on only one subcommittee?
(2) How many students in all are on carnival subcommittees?

[Part 41] [Page 11]

Check your answers.

Construction	Food	Games	Finance
Al	Don	Al	George
Bill	Hal	Hal	Julie
Don	Laura	Laura	Margie
Frank	Nancy	Pam	Sue

Tickets	Prizes	Side Show	Rides
George	Charles	Charles	[No one
Margie	Ed	Ed	volunteered.]
	Julie	Julie	
	Kathy	Kathy	

There are 10 students who are on more than one subcommittee.

(1) There are 5 students who are on only one subcommittee.
(2) There are 15 students in all on carnival subcommittees.

Record your results in the result-check column on your work sheet.

* * *

Since each of the subcommittees is really just a small committee, we shall refer to them as committees.

The Construction Committee and the Side Show Committee met together to discuss the booth for the Side Show.

Answer these questions on your work sheet.

(1) How many students are on the Construction Committee?
(2) How many students are on the Side Show Committee?
(3) Is there any student who is a member of both committees?
(4) If everyone attended who was supposed to, how many students were present at this joint meeting of the Construction Committee and the Side Show Committee?

On page 12, the idea of the union of two "overlapping" sets is introduced.

The exercises hint at the idea that the number of elements in the union of two sets might not be the same as the sum of the number of elements in one set and the number of elements in the other set.

[Part 41] [Page 12]

Check your answers.

Construction	Side Show
Al	Charles
Bill	Ed
Don	Julie
Frank	Kathy

(1) There are 4 students on the Construction Committee.

(2) There are 4 students on the Side Show Committee.

(3) Is there any student on both committees? No

(4) There were 8 students present at the meeting.

Record your results in the result-check column on your work sheet.

* * *

Games Committee: Al, Hal, Laura, Pam

Food Committee: Don, Hal, Laura, Nancy

At the first meeting of the Games Committee, each member wrote his name on a separate slip of paper. These four slips of paper were sent to Mr. Brown, the carnival sponsor. The next day, the Food Committee met and each member wrote his name on a separate slip of paper. These slips were sent to Mr. Brown.

Answer these questions on your work sheet.

(1) How many slips of paper did Mr. Brown receive in all from these two committees?

(2) If the Games Committee and the Food Committee meet together, how many students should be present?

On page 13, the bracketed remark following the answer for Exercise (2) is a beginning in the developing of the important idea that a given thing is either a member of a set, or it isn't. If it is a member, it is only one member regardless of how many times its name is listed.

* * *

The change from using names to using just initials is meant to do more than just simplify the listing of students. The programer felt that this would be a first step away from the concrete example of students and committees and a step toward the abstract ideas of numbers, ordered pairs of numbers, and sets of numbers.

The exercise is meant to familiarize students with the new notation.

[Part 41] [Page 13]

Check your answers.

Games: Al, Hal, Laura, Pam

Food: Don, Hal, Laura, Nancy

(1) Mr. Brown received 8 slips of paper, four from each committee.

(2) There should be 6 students present at a joint meeting of these two committees. [Even though Hal is on both committees, there is just one student named 'Hal'. The same is true of Laura.]

Record your results in the result-check column.

* * *

We said earlier that no two of the students who volunteered for committee work have the same first name. Not only that, but no two first names have the same first letter. So, from here on, we will list a student by using just the first letter in his [or her] first name. For example, in listing the Games Committee members [see above], we shall write:

Games: A, H, L, P

On your work sheet, list the members of the Food Committee by using only the first letter in each name. [The members of the Food Committee are listed near the top of this page.]

The exercises have two purposes; (1) to present an example of the union of a set with one of its proper subsets, and (2) to reinforce the idea that a given member of a set is counted as only one member, regardless of how many times its name is used.

On page 14, the bracketed remark following the answer hints at the idea that the order in which the members of a set are listed does not effect the membership of the set.

The text and exercise on page 15 call attention to the important word 'or', which is a key word in the definition of union.

[Part 41] [Page 14]

Check your answer.

Food: Don, Hal, Laura, Nancy

Using only the first letter in each student's name, the Food Committee can be listed as: D, H, L, N. [Of course, if you listed them in a different order, you are still right.]

Record your results in the result-check column.

* * *

The Finance Committee and the Tickets Committee held a joint meeting to discuss what to do with the money from the advance sale of tickets.

Here are lists of these two committees. [Of course, now we use only the first letter in a student's name.]

Finance: G, J, M, S

Tickets: G, M

Write the answers to these questions on your work sheet.

(1) Look at the lists of the two committees. In all, how many letter symbols are printed in the boxes?

(2) How many students should be present at the joint meeting?

Turn to PAGE 15.

[Part 41] [Page 15]

Check your answers.

Finance: G, J, M, S

Tickets: G, M

(1) There are 6 letter symbols in all printed in the boxes.

(2) 4 students should be present at the joint meeting.

Record your results in the result-check column.

* * *

You will remember that when the Games Committee and the Food Committee met together, there were 6 students who should have been present As a reminder, here are lists of these committees.

Games: A, H, L, P

Food: D, H, L, N

There is another way of telling someone that the combined membership consists of 6 students. Just say that there are 6 students in the group consisting of members of the Games Committee or the Food Committee.

Two committees are listed below.

Finance: G, J, M, S

Prizes: C, E, J, K

On your work sheet, tell how many students are in the group consisting of members of the Finance Committee or the Prizes Committee.

[Part 41] [Page 16]

Check your answer.

Finance: G, J, M, S

Prizes: C, E, J, K

There are 7 students in the group consisting of members of the Finance Committee or the Prizes Committee. [Since Julie is on both committees, a 'J' appears on both lists. However, there is only one student named 'Julie'.]

Record your results in the result-check column.

* * *

Mr. Brown, the carnival sponsor, called a joint meeting of the Prizes Committee and the Side Show Committee to decide whether prizes should be given at the side show.

Prizes: C, E, J, K

Side Show: C, E, J, K

Answer these questions on your work sheet.

(1) How many students are on the Prizes Committee or the Side Show Committee?

(2) Which students should attend the meeting called by Mr. Brown? [Use first letters to list students.]

The exercise on page 17 requires that the student copy his choice, not just recognize the correct choice. Students thus gain practice in writing the usual brace-notation name for a set.

[Part 41] [Page 17]

Check your answers.

Prizes: C, E, J, K

Side Show: C, E, J, K

(1) 4 students are on either the Prizes Committee or the Side Show Committee.

(2) C, E, J, and K should attend the meeting.

Record your results in the result-check column.

* * *

Each carnival subcommittee is a set of students. For example, the Prizes Committee is a set of students.

Which of these is the mathematical way to indicate that the students on the Prizes Committee are members of a set? Copy your choice onto your work sheet.

Prizes Committee = C, E, J, K

Prizes Committee = {C, E, J, K}

[Part 41] [Page 18]

ck your answer.

Prizes Committee = {C, E, J, K}

When we use braces in writing the name of a set, we say that we have written a brace-notation name for the set.

Record your result on your work sheet.

* * *

No one volunteered for the Rides Committee. Since we are now thinking of committees as sets of students, which of these is the mathematical way of expressing the idea that the Rides Committee is the set with no members?

Copy your choice onto your work sheet.

Rides Committee = ∅

Rides Committee = _

Rides Committee = 0

[Part 41] [Page 19]

Check your answer.

Rides Committee = ∅

['0' is not the correct answer for this exercise. Since '0' is a name for the number zero, and 'Rides Committee' is a name for one of the carnival committees, the sentence:

Rides Committee = 0

is nonsense. It says that a carnival committee is the same thing as a number!]

Record your results in the result-check column.

* * *

Mr. Brown called a meeting of all students who were on either the Construction Committee or the Rides Committee. [At that time, he didn't know that no one had volunteered for the Rides Committee.]

Do these exercises on your work sheet. [The Construction Committee is listed near the top of this page.]

(1) Use brace-notation to name the set of students who should have attended Mr. Brown's meeting.

(2) The set of students on either the Construction Committee or the Rides Committee is _____?_____. [Use brace-notation.]

The Booth Committee is introduced to help the development of the idea that the union of two sets is a set, and to pave the way for the introduction of the word 'union' on page 21.

[Part 41] [Page 20]

Check your answers.

Construction Committee = {A, B, D, F}

Rides Committee = ∅

(1) The set of students who should have attended Mr. Brown's meeting is {A, B, D, F}.

(2) The set of students on either the Construction Committee or the Rides Committee is {A, B, D, F}.

Record your results in the result-check column.

* * *

On the day before the carnival, someone finally realized that nothing was being done about a ticket booth. Something had to be done quickly, so a new committee was formed called the 'Booth Committee'. It was decided that the Booth Committee should consist of all students on either the Construction Committee or the Tickets Committee.

Construction Committee = {A, B, D, F}

Tickets Committee = {C, M}

Complete this on your work sheet [use brace-notation]:

Booth Committee = _____?_____

A strong hint is given in Exercise (1) on page 21. We want to be sure students answer correctly.

[Part 41] [Page 21]

Check your answer.

Construction Committee = {A, B, D, F}

Tickets Committee = {C, M}

The Booth Committee consists of all students on either of these committees. So,

Booth Committee = {A, B, D, F, G, M}.

You were correct as long as you named all of the elements of the Booth Committee, even if you wrote their names in a different order from that in the answer given above.

Record your results in the result-check column.

* * *

When two committees meet together or when two committees are combined to form another committee, we say that the joint committee is the union of the two committees.

So, for example, the Booth Committee is the union of the Construction Committee and the Tickets Committee.

Complete these sentences on your work sheet.

(1) The Booth Committee is the ___?___ of the Construction Committee and the Tickets Committee.

(2) {A, B, D, F, G, M} is the ___?___ of {A, B, D, F} and {C, M}.

On page 22, we begin to develop the need for a symbol to indicate the unioning operation. This is done by temporarily using the notation as shown in:

$$\{A, B, D, F, G, M\}$$
$$= \{A, B, D, F\} \text{ union } \{G, M\}$$

The alternative forms shown in the middle of page 22 are given to indicate that the two set names and the word 'union' can appear on either side of the equal sign. Also, the third alternative indicates that the name for the larger set does not necessarily appear first.

* * *

On page 25, the exercises are concerned with the union of two sets, each of which contains the number 999. Notice that the numeral '999' is an eye-catcher. This a subtle hint to students that 999 plays an important role in these exercises.

[Part 41] [Page 22]

Check your answers.

(1) The Booth Committee is the union of the Construction Committee and the Tickets Committee.

(2) {A, B, D, F, G, M} is the union of {A, B, D, F} and {G, M}.

Record your results in the result-check column.

* * *

Instead of writing:

{A, B, D, F, G, M} is the union of {A, B, D, F} and {G, M}

let's agree for now to shorten this by writing:

{A, B, D, F, G, M} = {A, B, D, F} union {G, M}

or:

{A, B, D, F} union {G, M} = {A, B, D, F, G, M}

or

{G, M} union {A, B, D, F} = {A, B, D, F, G, M}

Here are listed two of the carnival subcommittees.

Food Committee = {D, H, L, N}

Tickets Committee = {G, M}

Fill in these blanks on your work sheet.

(1) The students on the Food Committee or the Tickets Committee are _?_, _?_, _?_, _?_, _?_, _?_.

(2) The union of the Food Committee and the Tickets Committee = {___?___}.

(3) {D, H, L, N} union {G, M} = ___?___.

[Part 41] [Page 23]

Check your answers.

Food Committee = {D, H, L, N}

Tickets Committee = {G, M}

(1) The students on the Food Committee or the Tickets Committee are D, H, L, N, G, M.

(2) The union of the Food Committee and the Tickets Committee = {D, H, L, N, G, M}.

(3) {D, H, L, N} union {G, M} = {D, H, L, N, G, M}

[You are correct as long as you wrote these six letter symbols and no others.]

Record your results in the result-check column.

* * *

Mr. Brown posted a notice which said:

> There will be a meeting today at 3:30 of all students who are on either the Games Committee or the Food Committee.

Mr. Brown was pleased when he arrived at the meeting because all students were present who were supposed to be.

Games Committee = {A, H, L, P}

Food Committee = {D, H, L, N}

Do these exercises on your work sheet.

(1) How many students were present at the meeting?

(2) The union of the Games Committee and the Food Committee = {___?___}.

(3) {A, H, L, P} union {D, H, L, N} = ___?___

[Part 41] [Page 24]

Check your answers.

Games Committee = {A, H, L, P}

Food Committee = {D, H, L, N}

(1) 6 students were present at the meeting.

(2) The union of the Games Committee and the Food Committee = {A, H, L, P, D, N}. [Your answer is correct as long as you wrote these six letters, and no more. The order in which the elements of a set are listed does not change the membership of the set.]

(3) {A, H, L, P} union {D, H, L, N} = {A, H, L, P, D, N}.

Record your results on your work sheet.

* * *

We'll get back to the carnival a little later on, but now let's take a look at the union of two sets where sets of numbers are involved. [So far, we have considered the union of two sets where sets of students were involved.]

Here are two sets, A and B.

A = {1, 7}

B = {2, 19, 37}

Do these exercises on your work sheet.

(1) The set consisting of all the numbers in set A or set B is {___?___}.

(2) The union of {1, 7} and {2, 19, 37} = {___?___}.

(3) {1, 7} union {2, 19, 37} = ___?___

[Part 41] [Page 25]

Check your answers.

A = {1, 7}
B = {2, 19, 37}

(1) {1, 7, 2, 19, 37} is the set consisting of all the numbers in set A or set B.

(2) The union of {1, 7} and {2, 19, 37} is {1, 7, 2, 19, 37}.

(3) {1, 7} union {2, 19, 37} = {1, 7, 2, 19, 37}

Remember, you do not have to list the members of a set in any particular order, just so you include those [and only those] that should be included.

Record your results in the result-check column.

* * *

Now, let's try some examples where the same number is an element of both sets.

Do these exercises on your work sheet.

(1) List all numbers which are members of {3, 999} or {1, 7, 999}.

(2) {3, 999} union {1, 7, 999} ___?___

(3) How many numbers are there in the set which is the union of {3, 999} and {1, 7, 999}?

[Part 41] [Page 26]

Check your answers.

(1) The numbers which are members of {3, 999} or {1, 7, 999} are 1, 7, 3, and 999.

(2) {3, 999} union {1, 7, 999} {1, 7, 3, 999}

(3) There are 4 numbers in the set which is the union of {3, 999} and {1, 7, 999}.

In answering (3), we hope you didn't count 999 twice and write a '5' for your answer. After all, the number 999 is either a member of a set or it isn't. If it is a member of a set, we would count it only once in finding the number of elements in the set.

Record your results in the result-check column.

* * *

Suppose that A and B are sets. Instead of writing 'A union B', we can write 'A ∪ B'. ['A ∪ B' is read as 'A union B'.] '∪' is called 'the union sign'. [Notice that '∪' reminds one of the first letter in 'union'.]

So, for example, we shall write:

{1, 3} ∪ {3, 77, 88} = {1, 3, 77, 88}

Do these exercises on your work sheet.

(1) {1, 4, 9} ∪ {16, 25} = {___?___}

(2) {1, 3, 5} ___?___ {2, 4} = {1, 3, 5, 2, 4}

(3) Rewrite this sentence, using the example as a model:

{2, 7} union {7, 8, 9} is {2, 7, 8, 9}

The exercises on page 32 required that the members of sets A, B, and A ∪ B be listed.

* * *

The eight exercises on page 33 serve as a sort of self-test. Students who fail to answer all eight exercises correctly will be directed into a remedial loop for additional help.

Students who have a reasonably good grasp of the concept of union should have no trouble with Exercise (8), even though this is the first time they have been asked to give the union of two sets in which the members are different kinds of things.

[Part 41] [Page 33]

Check your answers.

A = {(x, y), x and y integers: 0 < x < 4 and y = 4}

B = {(z, w), z and w integers: z = 3 and 3 < w < 7}

(1) A = {(1,4), (2,4), (3,4)}

(2) B = {(3,4), (3,5), (3,6)}

(3) A ∪ B = {(1,4), (2,4), (3,4), (3,5), (3,6)}

Record your results on your work sheet.

* * *

Complete each of these exercises on your work sheet. Use brace-notation.

(1) {3, 4, 5, 13, 14} ∪ {4, 5} = ___?___

(2) {(2, 6), (5, 9)} ∪ {(−2, −6), (1, 4)} = ___?___

(3) {(3, 8), (4, 9), (5, 10)} ∪ ∅ = ___?___

(4) {9, 10, 11} ∪ {9, 10, 11} = ___?___

(5) {4, 9} ∪ {1, 2, 3} = ___?___

(6) {(−3, 5), (−2, 6), (−1, 7)} ∪ {(−3, 5)} = ___?___

(7) {(3, 6), (4, 8), (5, 10)} ∪ {(3, 6), (4, 7)} = ___?___

(8) {A, D, H} ∪ {1, 2, 3} = ___?___

[Part 41] [Page 34]

Check your answers.

(1) $\{3, 4, 5, 13, 14\} \cup \{4, 5\} = \{3, 4, 5, 13, 14\}$

(2) $\{(2, 6), (5, 9)\} \cup \{(-2, -6), (1, 4)\} = \{(2,6), (5,9), (-2,-6), (1,4)\}$

(3) $\{(3, 8), (4, 9), (5, 10)\} \cup \emptyset = \{(3,8), (4,9), (5,10)\}$

(4) $\{9, 10, 11\} \cup \{9, 10, 11\} = \{9, 10, 11\}$

(5) $\{4, 9\} \cup \{1, 2, 3\} = \{4, 9, 1, 2, 3\}$

(6) $\{(-3, 5), (-2, 6), (-1, 7)\} \cup \{(-3, 5)\} = \{(-3,5), (-2,6), (-1,7)\}$

(7) $\{(3, 6), (4, 8), (5, 10)\} \cup \{(3, 6), (4, 7)\} = \{(3,6), (4,8), (5,10), (4,7)\}$

(8) $\{A, D, H\} \cup \{1, 2, 3\} = \{A, D, H, 1, 2, 3\}$

Record your results on your work sheet.

If you got all the exercises above correct, turn to PAGE 38.

If you missed any of the exercises above, turn to PAGE 35.

Pages 35 - 37 constitute a remedial branch for the concept of union. We have tried to make the approach quite elementary.

The exercises on page 39 require application of the definition of union.

The primary purpose of these exercises is to prepare for the completely abstract exercises presented on page 41.

Answers to exercises on page 41.

(1) Yes
(2) No
(3) Yes
(4) Yes
(5) No

[Part 41] [Page 35]

Here is a way that might help you find the union of any two sets.

Let's suppose that

$A = \{1, 2, 3\}, \quad B = \{3, 4, 5\},$

and you wish to find $A \cup B$.

Imagine yourself going up to each element in set A and each element in set B and saying, "You are a member of the union." [Meaning, of course, A union B.] When you have spoken to every element in the sets, put them, and only them, all together in one set and you have $A \cup B$.

$\{1, 2, 3, 4, 5, 6\}$ is not $A \cup B$ because 6 is not a member of A or B and so, 6 was not one of the elements you told to be in the union.

$\{1, 2, 3, 4\}$ is not $A \cup B$ because 5 has been left out. [5 was one of the elements you told to be in the union.]

'$\{1, 2, 3, 3, 4, 5\}$' is a name for $A \cup B$, but it is not the best name since it lists 3 more than once. There is only one number 3, so if we list it more than once, we are doing more work than we need to do.

Complete this on your work sheet. Remember, use brace-notation.

$A \cup B =$ ___?___

[Part 41] [Page 36]

Check your answer.

$A \cup B = \{1, 2, 3, 4, 5\}$

Record your result on your work sheet.

* * *

Suppose that you wanted to find $\{5, 6, 7\} \cup \emptyset$. You would go up to 5 and say. "you are a member of the union". You would say the same to 6 and to 7. Next, speak to each element of $\emptyset$. Since there are no elements in $\emptyset$, the telling job is finished.

$\{5, 6, 7, \emptyset\}$ is not the union you are looking for because $\emptyset$ is not an element of either $\{5, 6, 7\}$ or $\emptyset$. [Remember, $\emptyset$ has no elements.]

Complete this on your work sheet.

$\{5, 6, 7\} \cup \emptyset =$ ___?___

[Part 41] [Page 37]

Check your answer.

$$\{5, 6, 7\} \cup \emptyset = \{5, 6, 7\}$$

Record your result on your work sheet.

* * *

Suppose that you wished to find the union of {M, N} and {4, 5, 6}. You would go up to M and say, "You are a member of the union". You would say the same thing to N. Then you would go up to 5, 6, and 7, and to each of them you would say, "You are a member of the union".

Now you would collect all of the elements you have told to be in the union and put them together in one set. If you do this, you get {M, N, 4, 5, 6}.

So, $\{M, N\} \cup \{4, 5, 6\} = \{M, N, 4, 5, 6\}$.

[Part 41] [Page 38]

Now it's time to state a definition of union. Complete this definition on your work sheet.

> Suppose that A and B are sets. Then, each element of $A \cup B$ belongs to set ___?___ or it belongs to set B; also each element of set A and each element of set ___?___ belongs to $A \cup B$.

[Part 41] [Page 39]

Check your answers.

Here is a definition of union.

> Suppose that A and B are sets. Then, each element of $A \cup B$ belongs to set A (1) or it belongs to set B; also each element of set A and each element of set B (2) belongs to $A \cup B$.

Record your results on your work sheet.

* * *

Answer these questions on your work sheet by writing a 'yes' or a 'no'.

Suppose that

$S = \{a, b, c\}$, and

$T = \{a, b, x, y, z\}$.

(1) Notice that c belongs to S. Does it follow that c belongs to $S \cup T$?

(2) Notice that x is a member of $S \cup T$ and x is a member of T. Does it follow that x is a member of S?

(3) Notice that z is a member of $S \cup T$ and z is not a member of S. Does it follow that z is a member of T?

(4) Notice that c is not a member of $S \cup T$. Does it follow that c is not a member of S?

(5) Notice that y does not belong to S. Does it follow that y does not belong to $S \cup T$?

[Part 41] [Page 41]

Answer these questions on your work sheet by writing a 'yes' or a 'no'. [Use 'no' to mean the same as 'not necessarily'.]

Suppose that C and D are sets.

(1) If u belongs to C, does it follow that u belongs to $C \cup D$?

(2) If v is a member of $C \cup D$ and v is a member of C, does it follow that v is a member of D?

(3) If w is a member of $C \cup D$ and w is not a member of C, does it follow that w is a member of D?

(4) If x is not a member of $C \cup D$, does it follow that x is not a member of C?

(5) If y does not belong to D, does it follow that y does not belong to $C \cup D$?

Beginning on page 44, Part 41 is devoted to developing the concept of intersection. The development is similar to that previously described in connection with the concept of union.

We present here just a few pages from the development for intersection to indicate the approach used.

At this point, we are able to take advantage of the previously introduced set-notation for the various committees.

[Part 41] [Page 44]

Now, let's get back to the preparations for the carnival at Zabranchburg High School.

Mr. Brown, the carnival sponsor, posted a notice which said:

> Today, at 3:30, the Food Committee will meet in Room 217. The Games Committee will meet at the same time in Room 111.

After reading this notice, one student was heard to remark, "I can't be in two places at the same time."

The two committees are given below.

Food Committee = {D, H, L, N}

Games Committee = {A, H, L, P}

List on your work sheet the students who might have made the remark quoted above.

[Part 41] [Page 45]

Check your answers.

Food Committee = {D, H, L, N}

Games Committee = {A, H, L, P}

H or L might have made the remark.
(a) (b)

Record your results on your work sheet.

* * *

Here are various committees listed in pairs.

(1) Tickets Committee = {G, M}
Finance Committee = {G, J, M, S}

(2) Prizes Committee = {C, E, J, K}
Side Show Committee = {C, E, J, K}

(3) Food Committee = {D, H, L, N}
Finance Committee = {C, J, M, S}

(4) Games Committee = {A, H, L, P}
Rides Committee = ∅

Suppose that for each pair, the two committees were scheduled to meet at the same time, but in different rooms.

For each pair of committees, use brace-notation to name the set of students who were supposed to be in two places at the same time. [Remember, use a '∅' to name the set with no members.]

Write your answers on your work sheet.

[Part 41] [Page 46]

Check your answers.

	Set of students on both committees
(1) Tickets Committee = {C, M} Finance Committee = {G, J, M, S}	(1) {G, M}
(2) Prizes Committee = {C, E, J, K} Side Show Committee = {C, E, J, K}	(2) {C, E, J, K}
(3) Food Committee = {D, H, L, N} Finance Committee = {C, J, M, S}	(3) ∅
(4) Games Committee = {A, H, L, P} Rides Committee = ∅	(4) ∅

Record your results on your work sheet.

* * *

Take another look at Exercise (1) above. Suppose that the two committees listed there are scheduled to meet at the same time, but in different rooms.

Instead of saying that {C, M} is the set of students who are supposed to be in two places at the same time, we can say that {C, M} is the set of students who are members of both the Tickets Committee and the Finance Committee.

On your work sheet, give a brace-notation name for the set of students who are members of both the Food Committe and the Games Committee.

Food Committee = {D, H, L, N}

Games Committee = {A, H, L, P}

Part 42 gives students extensive practice in plotting sets of ordered pairs of integers. Both finite sets and infinite sets are plotted. In addition, the concepts of union and intersection are extended, and this theorem is presented:

For each set A, for each set B,
$n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

['n(X)' means the same as 'the number of elements in set X'.]
We present here a selected sample of pages from Part 42.

[Part 42] **[Page 1]**

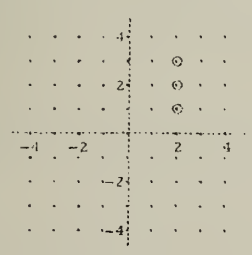
Fill in the heading on your work sheet.

You learned earlier that a set of ordered pairs of numbers is often called 'a set of points'.

You also learned that a set of points can be plotted on a picture of the number plane lattice.

Sample. Plot the points in set A if A = {(2, 1), (2, 2), (2, 3)}.

Solution.



We have drawn a loop around each dot which corresponds to a point in set A.

In this book, we shall always plot the points in any set called 'A' by drawing a loop around each dot corresponding to a point in set A. It will be easier for you to compare your graphs to ours if you will follow the same convention in plotting points in sets called 'A'.

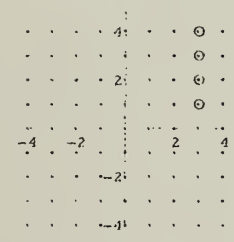
Plot this set of points on your work sheet.

A = {(x, y), x and y integers: x = 3 and 0 < y < 5}

[Part 42] [Page 2]

Check your answer.

A = {(x, y), x and y integers: x = 3 and 0 < y < 5}



Record your result on your work sheet.

* * *

In addition to sets called 'A', you will be working with sets called 'X'. We shall plot the points in any set called 'X' by drawing small cross-marks over the appropriate dots. | X |

Use the picture of the number plane lattice on your work sheet and plot this set of points.

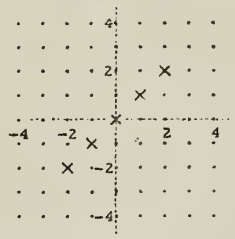
X = {(x, y), x and y integers: x = y and −3 < x < 3}

[Hint. Set X consists of five points.]

[Part 42] [Page 3]

Check your answer.

$X = \{(x, y),\ x \text{ and } y \text{ integers: } x = y \text{ and } -3 < x < 3\}$



Record your result on your work sheet.

* * *

Do these exercises on your work sheet.

$A = \{(x, y),\ x \text{ and } y \text{ integers: } x = y \text{ and } -3 \leq x \leq 3\}$

$X = \{(m, n),\ m \text{ and } n \text{ integers: } 0 < m < 3 \text{ and } -1 < n < 3\}$

(1) Plot the points in each of these sets on the <u>same</u> picture of the number plane lattice. [Remember, loops for A, cross-marks for X.]

(2) How many points are in set A?

(3) How many points are in set X?

Pages 21 - 24 are preparation for the exercises on page 25. [Previous to page 21, students learned that 'n(X)' means the number of elements in set X.]

[Part 42] [Page 21]

Suppose you were asked to list the elements of the set

$\{(x, y),\ x \text{ and } y \text{ integers: } x + y < 2\}$.

You might start out something like this:

(0, 1), (0, 0), (0, −1), (0, −2), (−473, 474), (16, −928), ...

Soon you would realize that it is impossible for anyone to list all elements of the set where the only condition required for an ordered pair (x, y) of integers to be a member of the set is that it satisfy the sentence '$x + y < 2$'.

But suppose more conditions are added. For example, suppose you were asked to list the elements of the set

$\{(x, y),\ x \text{ and } y \text{ integers: } x + y < 2,\ x \geq 0, \text{ and } y \geq 0\}$.

This is easy. The elements are (0, 0), (0, 1), and (1, 0).

Now consider set R.

$R = \{(x, y),\ x \text{ and } y \text{ integers: } \underbrace{\overset{(a)}{x + y \leq 3},\ \overset{(b)}{x \geq 0}, \text{ and } \overset{(c)}{y \geq 0}}_{\text{conditions}}\}$.

For each ordered pair listed below, write 'yes' in the blank if it is an element of R. If the ordered pair is not an element of R, tell which condition is not met by writing either 'a', 'b', or 'c' in the blank. Write your answers on your work sheet.

(1) (2, 0) __?__ (2) (5, 27) __?__ (3) (−4, 4) __?__

(4) (0, 3) __?__ (5) (1, 2) __?__ (6) (5, −4) __?__

[Part 42] [Page 22]

Check your answers.

$R = \{(x, y),\ x \text{ and } y \text{ integers: } x + y \leq 3,\ x \geq 0, \text{ and } y \geq 0\}$

conditions ⟶ (a) (b) (c)

(1) (2, 0) yes (2) (5, 27) a (3) (−4, 4) b

(4) (0, 3) yes (5) (1, 2) yes (6) (5, −4) c

Record your results on your work sheet.

* * *

Set R is named near the top of this page. On your work sheet, give another name for set R by listing its elements. That is, complete this:

R = { __?__, __?__, __?__, __?__, __?__, __?__, __?__, __?__, __?__, __?__ }.

Check your answer.

$R = \{(x, y),\ x \text{ and } y \text{ integers: } x + y \leq 3,\ x \geq 0, \text{ and } y \geq 0\}$

$R = \{(0,0), (0,1), (0,2), (0,3), (1,0), (1,1), (1,2), (2,0), (2,1), (3,0)\}$

Record your results on your work sheet.

* * *

Now, consider the set S where

$$S = \{(x, y),\ x \text{ and } y \text{ integers: } \underbrace{\underset{(a)}{x + y > 3},\ \underset{(b)}{x < 4}, \text{ and } \underset{(c)}{y < 4}}_{\text{conditions}}\}.$$

For each ordered pair listed in Exercises (1) - (6), write 'yes' in the blank if it is an element of S. If the ordered pair is not an element of S, tell which condition is not met by writing either 'a', 'b', or 'c' in the blank. Write your answers on your work sheet.

(1) (1, 3) ___?___ (2) (3, 1) ___?___ (3) (0, 4) ___?___

(4) (4, 0) ___?___ (5) (3, 3) ___?___ (6) (1, 2) ___?___

Check your answers.

$S = \{(x, y),\ x \text{ and } y \text{ integers: } x + y > 3,\ x < 4, \text{ and } y < 4\}$

conditions ⟶ (a) (b) (c)

(1) (1, 3) ___yes___ (2) (3, 1) ___yes___ (3) (0, 4) ___c___

(4) (4, 0) ___b___ (5) (3, 3) ___yes___ (6) (1, 2) ___a___

Record your results on your work sheet.

* * *

The set S referred to in the following exercises is the set S named near the top of this page. Notice that the ordered pairs named in Exercises (1) (2), and (5) above are some of the members of set S.

Do these exercises on your work sheet.

(1) S = {___?___, ___?___, ___?___, ___?___, ___?___, ___?___}

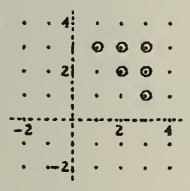
(2) Plot the points in set S.

Check your answers.

$S = \{(x, y),\ x \text{ and } y \text{ integers: } x + y > 3,\ x < 4, \text{ and } y < 4\}$

(1) $S = \{(1,3), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

(2)



Record your results on your work sheet.

* * *

Do these exercises on your work sheet.

$A = \{(x, y),\ x \text{ and } y \text{ integers: } x + y \leq 5,\ x \geq 0, \text{ and } y \geq 0\}$

$X = \{(x, y),\ x \text{ and } y \text{ integers: } x + y > 5,\ x < 6, \text{ and } y < 6\}$

(1) Plot the points in each set on the same diagram.

(2) $n(A) =$ ___?___ (3) $n(X) =$ ___?___ (4) $n(A \cap X) =$ ___?___

(5) $n(A \cup X) =$ ___?___

[Try to answer (5) just by using your answers to (2), (3), and (4).]

Pages 31 - 37 contain a detailed look at the theorem:

For each set A, for each set B,
$n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

for students that could not correctly fill all the blanks in the table below.

n(A)	5	8	10	9	87
n(X)	0	10	18	21	43
$n(A \cap X)$	0	0	8	6	16
$n(A \cup X)$	?	?	?	?	?

[Part 42] [Page 31]

Perhaps you are wondering why it is not always possible to find the number of elements in the union of two sets simply by adding the number of elements in the first set and the number of elements in the second set.

Let's look at a few examples which might help you understand how you can find the number of elements in the union of two sets without actually finding or counting the elements in the union.

Suppose that

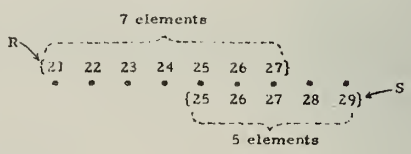
$R = \{21, 22, 23, 24, 25, 26, 27\}$ and

$S = \{25, 26, 27, 28, 29\}$.

By counting the elements in each set, we find that

$n(R) = 7$ and $n(S) = 5$.

Now, look at the following diagram.



It is easy to see that $n(R \cup S) = 9$. [Just count the dots.] Now, since $9 \neq 7 + 5$, $n(R \cup S) \neq n(R) + n(S)$. That is, you don't get $n(R \cup S)$ just by adding $n(R)$ and $n(S)$.

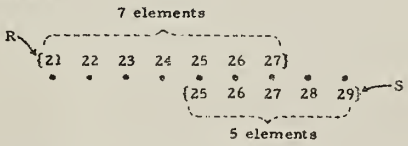
Answer these exercises on your work sheet.

(1) Just by adding n(R) and n(S), you get a total of 12. And, 12 is __7__ more than $n(R \cup S)$.

(2) How many elements do R and S have in common? That is, what is $n(R \cap S)$?

[Part 42] [Page 32]

Check your answers.



(1) Just by adding n(R) and n(S), you get a total of 12. And, 12 is __3__ more than $n(R \cup S)$.

(2) The sets R and S have __3__ elements in common. That is, $n(R \cap S) =$ __3__.

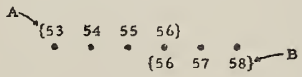
Record your results on your work sheet.

* * *

Suppose that

$A = \{53, 54, 55, 56\}$ and

$B = \{56, 57, 58\}$.



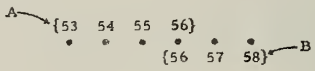
Do these exercises on your work sheet.

(1) n(A) = __?__ (2) n(B) = __?__ (3) $n(A \cup B)$ = __?__

(4) n(A) + n(B) is __7__ more than $n(A \cup B)$. (5) $n(A \cap B)$ = __7__

[Part 42] [Page 33]

Check your answers.



(1) n(A) = __4__ (2) n(B) = __3__ (3) $n(A \cup B)$ = __6__

(4) n(A) + n(B) is __1__ more than $n(A \cup B)$. (5) $n(A \cap B)$ = __1__

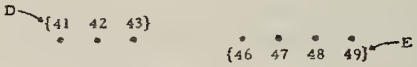
Record your results on your work sheet.

* * *

Suppose that

$D = \{41, 42, 43\}$ and

$E = \{46, 47, 48, 49\}$.



Do these exercises on your work sheet.

(1) n(D) = __?__ (2) n(E) = __?__ (3) $n(D \cup E)$ = __?__

(4) n(D) + n(E) is __?__ more than $n(D \cup E)$. (5) $n(D \cap E)$ = __?__

[Part 42] [Page 34]

Check your answers.

D → {41 42 43}

{46 47 48 49} ← E

(1) n(D) = 3 (2) n(E) = 4 (3) n(D ∪ E) = 7

(4) n(D) + n(E) is 0 more than n(D ∪ E). (5) n(D ∩ E) = 0

Record your results on your work sheet.

* * *

Suppose that

F = {72, 73, 74, 75, 76} and

G = {72, 73, 74, 75, 76}.

F → {72 73 74 75 76}

{72 73 74 75 76} ← G

Do these exercises on your work sheet.

(1) n(F) = ? (2) n(G) = ? (3) n(F ∪ G) = ?

(4) n(F) + n(G) is ? more than n(F ∪ G). (5) n(F ∩ G) = ?

[Part 42] [Page 35]

Check your answers.

F = {72, 73, 74, 75, 76}

G = {72, 73, 74, 75, 76}

(1) n(F) = 5 (2) n(G) = 5 (3) n(F ∪ G) = 5

(4) n(F) + n(G) is 5 more than n(F ∪ G). (5) n(F ∩ G) = 5

Record your results on your work sheet.

* * *

Do these exercises on your work sheet.

Suppose that

J = {−22, −21, −20, −19} and

K = {−20, −19, −18, −17, −16}.

(1) n(J) = ? (2) n(K) = ? (3) n(J ∩ K) ?

(4) n(J ∪ K) = ?

(5) Is n(J) + n(K) the same as n(J ∪ K)? [Yes or no?]

(6) Is n(J) + n(K) −2 the same as n(J ∪ K)? [Yes or no?]

[Part 42] [Page 36]

Check your answers.

J = {−22, −21, −20, −19} and

K = {−20, −19, −18, −17, −16}.

(1) n(J) = 4 (2) n(K) = 5 (3) n(J ∩ K) = 2

(4) n(J ∪ K) = 7

(5) No [n(J ∪ K) ≠ n(J) + n(K)]

(6) Yes [n(J ∪ K) = n(J) + n(K) −2]

Record your results on your work sheet.

* * *

Do this exercise on your work sheet.

Suppose that P and Q are sets such that

n(P) = 10,

n(Q) = 8, and

n(P ∩ Q) = 5.

n(P ∪ Q) = 7

[Part 42] [Page 37]

Check your answer.

n(P) = 10

n(Q) = 8

n(P ∩ Q) = 5

n(P ∪ Q) = 13 [10 + 8 − 5 = 13.]

Record your result on your work sheet.

'N. F.' as used in the answers on page 69 means that the set in question is not finite. Thus, it is not possible to indicate the number of elements in the set.

* * *

Exercise (1) on page 69 is an example of a device which is frequently used in Part 42 to help students get started in plotting sets of points.

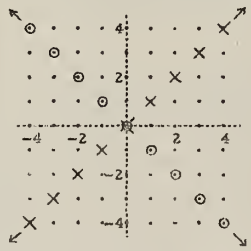
[Part 42] [Page 69]

Check your answers.

A = {(x, y), x and y integers: x + y = 0}

X = {(x, y), x and y integers: x - y = 0}

(1)



(2) n(A) = ∞

(3) n(X) = ∞

(4) n(A ∩ X) = 1

(5) n(A ∪ X) = ∞

Record your results on your work sheet.

* * *

Do these exercises on your work sheet.

A = {(x, y), x and y integers: y = |x|}

(1) Complete each of these to name a member of A.

(a) (1, ?) (b) (-1, ?) (c) (0, ?)

(d) (2, ?) (e) (-2, ?) (f) (3, ?)

(2) Does A contain any ordered pairs whose second components are negative? [If you think it does, try to find one.]

(3) Plot the points in set A.

[Part 42] [Page 70]

Check your answers.

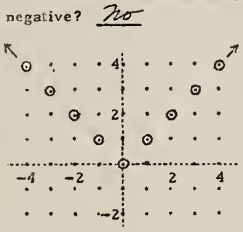
A = {(x, y), x and y integers: y = |x|}

(1) (a) (1, 1) (b) (-1, 1) (c) (0, 0)

(d) (2, 2) (e) (-2, 2) (f) (3, 3)

(2) Does A contain any ordered pairs whose second components are negative? no

(3)



Record your results on your work sheet.

* * *

Consider sets A and X.

A = {(x, y), x and y integers: y = x + 2}

X = {(x, y), x and y integers: y = |x| + 2}

Here is a partially completed table giving some of the members of sets A and X. Complete the table on your work sheet.

Set A	(0, 2)	(1, 3)	(2, ?)	(3, ?)	(-1, 1)	(-2, 0)	(-3, ?)
Set X	(0, 2)	(1, ?)	(2, ?)	(3, ?)	(-1, 3)	(-2, ?)	(-3, ?)

[Part 42] [Page 71]

Check your answers.

A = {(x, y), x and y integers: y = x + 2}

X = {(x, y), x and y integers: y = |x| + 2}

Some of the members of sets A and X are listed here.

Set A	(0, 2)	(1, 3)	(2, 4)	(3, 5)	(-1, 1)	(-2, 0)	(-3, -1)
Set X	(0, 2)	(1, 3)	(2, 4)	(3, 5)	(-1, 3)	(-2, 4)	(-3, 5)

Record your results on your work sheet.

* * *

Use the answers given above to help you plot the points in sets A and X on the same diagram. Do this on your work sheet. [Sets A and X are named near the top of this page.]

Exhibit F

Exhibit F

November 1962

ANNOUNCEMENT

The University of Illinois Committee on School Mathematics [UICSM] will hold a conference on the role of applications of mathematics in the secondary school mathematics curriculum. Dates for the conference are February 14 through February 19, 1963. The conference will be at Allerton House in Monticello, Illinois. [The enclosed brochure describes the meeting place.]

The UICSM has already produced and published, during the period 1951-1962, textbooks for students and teachers in the 9-12 grade range. We are now undertaking a completely new project for grades 7-12 with the financial support of the National Science Foundation. This project will involve considerable experimentation in the presentation of new subject matter and in radical reorganizations of "old" subject matter. In particular, we plan to make a serious study of the problems involved in incorporating applications of mathematics in a secondary school program. Some of the questions which present themselves are:

(1) Should basic mathematical concepts be drawn from applications, or should applications serve only to illustrate the concepts?

(2) Must applications be practical?

(3) What are the factors which influence the successful teaching of various applications? [Grade level, vocational interests, science background?]

(4) Should the mathematics program assume the responsibility for teaching the subject matter of the context of an application, or should an application wait until students have acquired the relevant background in other courses?

We seek advice on these matters. Probably, our greatest need is for examples of content which might be incorporated in instructional materials for experimental use at various grade levels. This experimentation should help us decide questions of feasibility and prerequisites, and also give us opportunities to develop pedagogical approaches. The project staff hopes to begin developing classroom materials based on the results of the conference, and these materials will be used experimentally starting in fall semester of 1963.

The conference will be conducted along very informal lines. We hope to have about fifteen guests who will meet with representatives from our staff. Some of the conferees will present papers which will serve as foci for general discussions; but, at least half of our working time will be spent in small and informal sessions. [At the present writing we have received acceptances for papers from Stephen Diliberto, Fred Mosteller, Henry Pollak, Arnold Ross, and Joachim Weyl. Richard Bellmann cannot attend but will send a paper.]

It is hoped that the conferees will also advise us on how to continue to utilize the talents of applied mathematicians and scientists as we proceed in our work.

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Questionnaire for 1962 High School Graduates

This questionnaire is an important part of a nationwide study of the mathematical training of 1962 high school graduates.

The study is being conducted by the University of Illinois Committee on School Mathematics in hopes that the findings may be used by school systems and individuals to strengthen the teaching of mathematics. This work is supported under a two-year grant from the National Science Foundation. You have been selected as one of the participants in this study. Your high school will be interested in the results of this follow-up study, as one source of data for evaluating its course offerings. It is of great importance that your questionnaire responses and permission slip for college transcripts be completed and returned promptly. It is hoped that you will do this now and return them in the self-addressed, stamped envelope that we have provided. According to a pilot study of these questionnaire items, this should take you about 15 minutes. We have purposely designed the study so as to hold your time expenditure to a minimum and thereby maximize returns from the students involved.

In the analysis of the data and writing of the report, students will have complete anonymity. We have coded the questionnaire forms both to eliminate reference to particular student names and to aid us in the follow-up of non-returned questionnaires.

Thank you for your cooperation.

Robert Comley
Research Assistant
UICSM Mathematics Project
1210 W. Springfield
Urbana, Illinois

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Directions:

1. Sign the permission slip that allows us to obtain your college transcripts.

2. Refer to the items of the inventory.

3. Fill in the information asked for in items 2, 3, and 4.

4. For items 5-44, indicate your feelings as follows:

If you	strongly disagree - circle A disagree - circle B have no opinion - circle C agree - circle D strongly agree - circle E		to the right of each item.

The Inventory Items

1. Student I.D. Number _____

2. Sex: Male _____ Female _____

3. High School _____

4. College _____

5.	I like mathematics as much or more than any of my other studies.	A B C D E
6.	Mathematicians are free to develop any self-consistent system they choose, regardless of whether or not they start with facts taken from experience.	A B C D E
7.	My ability to use the methods of mathematics to solve problems is equal to or surpasses that of many of my previous classmates.	A B C D E
8.	I don't ever want to study any more mathematics in the future.	A B C D E
9.	Mathematical systems, correctly perceived, provide aesthetic satisfaction akin to that of a beautiful painting or other work of art.	A B C D E

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strongly disagree	- circle A
disagree	- circle B
have no opinion	- circle C
agree	- circle D
strongly agree	- circle E

10. I like to read books about mathematics for some of my leisure reading.	A	B	C	D	E
11. The study of mathematics tends to curb the curiosity and imagination of students.	A	B	C	D	E
12. Mathematics is mainly fixed and unchangeable in character.	A	B	C	D	E
13. The main reason I took mathematics in high school was because it was required for college.	A	B	C	D	E
14. Mathematics is the science of quantity.	A	B	C	D	E
15. I cannot understand mathematics enough to apply it to problems I meet.	A	B	C	D	E
16. I expect to learn more mathematics in the future.	A	B	C	D	E
17. No external verification of definitions or axioms is required in mathematics, as long as it is considered only as mathematics.	A	B	C	D	E
18. I do extra reading or work extra problems for the mathematics courses which I study.	A	B	C	D	E
19. In high school, I disliked mathematics more than any other subject.	A	B	C	D	E
20. In the teaching of mathematics, the intuition of the student should first be employed and developed before the concepts are formally verbalized.	A	B	C	D	E
21. Mathematics may be compared to a great tree, ever putting forth green shoots and new branches.	A	B	C	D	E
22. Mathematics is essentially a language, unique in its ability for clarifying logical arguments.	A	B	C	D	E
23. In college, I disliked mathematics more than any other subject.	A	B	C	D	E
24. In discussing mathematics with classmates I prefer that we develop the conversation on an intuitive basis rather than being concerned with using careful language and statements.	A	B	C	D	E

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strongly disagree	- circle A
disagree	- circle B
have no opinion	- circle C
agree	- circle D
strongly agree	- circle E

25. Mathematics is essentially a set of techniques for solving problems.	A	B	C	D	E
26. I feel that I am more enthusiastic about mathematics than many of my classmates were who shared my mathematics classes.	A	B	C	D	E
27. After completing my high school courses, my general feeling, as related to mathematics, could best be described as confused.	A	B	C	D	E
28. I like to study mathematics.	A	B	C	D	E
29. I believe that I could invent some new mathematics over an extended period of time.	A	B	C	D	E
30. Among the social science courses, I liked best those that were mathematically oriented.	A	B	C	D	E
31. I don't like mathematics very much.	A	B	C	D	E
32. My mathematics courses have made me curious about many new ideas.	A	B	C	D	E
33. I would like to learn more mathematics in the future.	A	B	C	D	E
34. I can be successful without mathematics.	A	B	C	D	E
35. I enjoy doing mathematical games, puzzles, or brain teasers.	A	B	C	D	E
36. Mathematics is my most difficult course.	A	B	C	D	E
37. Mathematics fascinates me.	A	B	C	D	E
38. In college I enjoyed my mathematics classes.	A	B	C	D	E
39. Mathematics aims mainly at power of application of execution.	A	B	C	D	E
40. I feel a little frightened or anxious whenever I have to do something related to mathematics.	A	B	C	D	E
41. I like to wander freely about in topics in mathematics that are not required in my mathematics courses.	A	B	C	D	E

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strongly disagree	- circle A
disagree	- circle B
have no opinion	- circle C
agree	- circle D
strongly agree	- circle E

42. Mathematical theorems, like automobiles, may become obsolete because new and better ones are discovered.	A B C D E
43. I took my college mathematics courses because I enjoy learning mathematics.	A B C D E
44. I wish I had been allowed more electives in my college curriculum so that I could have taken more mathematics.	A B C D E

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Permission slip for college transcripts

The University of Illinois School Mathematics Project (UICSM) has my permission to obtain transcripts of my college records, including the grades for the present term.

_____ _____

(signature) [ ] (date)

1210 West Springfield · Urbana, Illinois 61801 · Telephone 333-0150 (Area Code 217)

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UNIVERSITY OF ILLINOIS COMMITTEE ON SCHOOL MATHEMATICS

UICSM

MATHEMATICS PROJECT

Max Beberman, Director

May 9, 1966

Dear Student:

The University of Illinois Committee on School Mathematics (UICSM) is conducting a follow-up study of 1962 high school graduates who went to college. Two weeks ago we mailed to you a questionnaire and a request that you allow us to obtain a copy of your college transcript. As yet, we have not received your reply. No doubt, the time pressure on you at this stage of the school year has prevented you from responding. However, we hope you can find time to fill out a much shorter version of the original questionnaire and to authorize our obtaining your college record. The shortened inventory of 16 items is enclosed, along with a permission slip for the college transcript. It will take less than 5 minutes to complete these forms and return them to us in the self-addressed envelope. These few minutes that you invest will contribute to bettering mathematics education at the high school level and will be greatly appreciated by us.

If you have recently mailed to us the previous questionnaire and permission slip, please ignore this letter and accept our apologies for troubling you to read it!

Sincerely,

Robert Comley

Robert Comley
Research Assistant

RC:ls

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Student Questionnaire--Short Form

Please circle the letter which best indicates your feelings about the items below, according to the categories: A—strongly disagree. B—disagree. C—have no opinion. D—agree. E—strongly agree.

1. Student I.D. Number _____

2. Sex: Male _____ Female _____

3. High School _____

4. College _____

5.	I like to read books about mathematics for some of my leisure reading.	A B C D E
6.	Mathematicians are free to develop any self-consistent system they choose, regardless of whether or not they start with facts taken from experience.	A B C D E
7.	I wish I had been allowed more electives in my college curriculum so that I could have taken more mathematics.	A B C D E
8.	I like to study mathematics.	A B C D E
9.	Among the social sciences courses, I liked best those that were mathematically oriented.	A B C D E
10.	I would like to learn more mathematics in the future.	A B C D E
11.	I took my college mathematics courses because I enjoy learning mathematics.	A B C D E
12.	No external verification of definitions or axioms is required in mathematics, as long as it is considered only as mathematics.	A B C D E
13.	I like mathematics as much or more than any of my other subjects.	A B C D E
14.	Mathematics is essentially a set of techniques for solving problems.	A B C D E
15.	I like to wander freely about in topics in mathematics that are not required in my mathematics courses.	A B C D E
16.	I do extra reading or work extra problems for the mathematics courses which I study.	A B C D E

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Permission slip for college transcripts

The University of Illinois School Mathematics Project (UICSM) has my permission to obtain transcripts of my college records, including the grades for the present term.

_____ _____

(signature) [ ] (date)

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